Technical

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A GENERALIZED NARROW-GROOVE THEORY FOR THE GAS-LUBRICATED HERRINGBONE THRUST BEARING

by

H. G. Elrod, Jr.

October 1968

Prepared under

Contract Nonr-2342(00)
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ABSTRACT

A narrow-groove theory for gas on liquid-lubricated herringbone thrust bearings is developed by means of two matched asymptotic expansions. The first expansion, for the film interior, yields a generalized Whipple equation for the average pressure level. The second expansion, for the film edges, yields a generalized Muijdermann-Body pressure correction. Arbitrary transverse groove shape is accommodated by the analysis.

The prognosis for development along present lines of a single partial differential equation to include first-order groove-width effects, both in the film interior and at its edges, is very good.

TABLE OF CONTENTS

LIST OF FIGURES

TABLE OF SYMBOLS
(Equation Number Indicates Where Symbol First Used)

A_j	Fourier coefficients, [8-1].
c	Nominal bearing clearance, [5].
C_1, C_2	Film-thickness functions, [37] and [38].
f	"function of"; deviational film-thickness functions, [67].
F	Pressure "ripple" function, [31]; see Appendix A.
$\mathcal{F}$	Film-thickness integral, [82].
G	Pressure "ripple" function, [31]; See Appendix A.
$\mathcal{G}$	Film-thickness integral, [82].
h	Local film thickness, [3].
H	Dimensionless film thickness, h/c, [8].
K_j	Film-thickness integrals, [29]; See Appendix A.
$\mathcal{K}$	Combination of film-thickness integrals, [151].
L	Shortest distance through bearing, y-direction, Fig. 1.
$\dot{m}$	Film mass flux per unit transverse width, [10].
$\dot{\mathcal{M}}$	Non-dimensional mass throughput per groove-ridge combination [12].
p	Local fluid press, [3]; subscript "a" denotes ambient.
q	Residual pressure function, [52].
$\tilde{q}$	Residual pressure function, [60].
R	Perfect-gas constant, [10]; film-thickness integral, [81]; See Appendix A.
s	Dummy variable, [100].
St	Sawtooth function, [57].
t	Dummy variable, [100].

T Absolute temperature [10].

U Surface velocity in x-direction, [3].

w Mapping function, [163].

W Load per groove-ridge cycle, [131].

x Distance along bearing edge, [1]; position in complex plane, [164].

y Distance normal to bearing edge, [1]; position in complex plane, [164].

z Distance normal to groove-ridge interface, [1]; complex variable, x + iy.

α First-order dimensionless pressure level correction, [25].

β Groove angle, [1]. See Fig. 1.

γ ζ-Position of groove-ridge interface.

δ Amplitude parameter, [67]; dirac delta-function.

Δ Groove-ridge wavelength, measured in z-direction, Fig. 1.

ϵ Groove-ridge wavelength parameter (small) = Δ/L

ζ Non-dimensional distance normal to groove-ridge interface, z/Δ, [5].

η Non-dimensional distance normal to bearing edge, $\xi/\epsilon = y/\Delta$, [46].

Λ Modified compressibility number, $\sin\beta \; 6\mu UL/p_a c^2$ [5].

Λ_{eff} Effective compressibility number, [28].

μ Viscosity, [3].

ξ Non-dimensional distance in y-direction, y/L, [5].

π Non-dimensional pressure, p/p_a, [5]; also, 3.14159...

π^* Shifted non-dimensional pressure, [44].

ψ π^2, [7]: also Diagmma function, $\Gamma'(x)/\Gamma(x)$, [64].

INTRODUCTION

There is today considerable interest in bearings employing grooved surfaces. Especially in the case of gas-lubricated bearings, grooving adds to load-carrying capacity and can provide improved stability characteristics. New applications have quickened interest in the corresponding theory of operation, and a number of theoretical papers have appeared in recent years on spiral-groove and herringbone bearings.

The first satisfactory published theory for grooved-plate bearings was that of Whipple(1) in 1951. He treated both gases and liquids. Parallel-plate geometry was assumed with straight parallel, rectangular grooving. In the narrow-groove limit, Whipple assumed linear pressure development transverse to the grooving. Then employing the principles of mass and pressure continuity, he eliminated the short-wavelength pressure ripple in favor of the general change of pressure associated with a complete grooving cycle (groove-ridge pair). A differential equation for the average pressure resulted. In effect, for gases the Whipple theory includes the long-range density changes associated with inlet and outlet pressures, but invokes local incompressibility across the grooving.

Various investigators explored the consequences of the

Whipple theory, but the next major improvement was that of Vohr and Pan[(2)] in 1963, who adapted the Whipple treatment to non-parallel plates with curved, non-parallel rectangular grooving. The Vohr-Pan differential equation substitutes for the usual Reynolds' equation when grooving is present, and it has been the basis of a number of design studies.

In the narrow-groove limit, the Vohr-Pan treatment becomes exact,* but with finite groove-width, deficiencies appear. The first of these deficiencies arises only with gases, and is due to the neglect of transverse compressibility effects. It is manifest in-the-large by the theoretical prediction that, even for gases, the load-carrying capacity of grooved-surface bearings is always proportional to the speed. Since this prediction is at variance with usual gas bearing performance, Wildmann[(3)] undertook a study of compressibility effects on a parallel-plate herringbone with slight sinusoidal undulations. This particular bearing geometry was chosen because the only required mathematical approximation (beyond Reynolds equation) is that of a well-accepted expansion in film-thickness excursion. Wildmann showed that the load capa-

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*Subject, of course, to the validity of Reynolds equation.

city of the herringbone first rises linearly with speed (as predicted by Whipple theory), reaches a maximum, and then diminishes to an asymptotic limit predictable on the assumption that (ph) is constant across the grooving. He also obtained the limiting expression for straight grooving of arbitrary depth and transverse section.

The second effect of finite groove-width is that of correspondingly finite pressure-ripples across the grooving. (Obviously, at fixed speed, the amplitude of these ripples tends to zero with groove-width). Compressibility aside, these finite ripples do not permit accommodation of predicted pressure distributions to edge boundary-conditions of uniform ambient pressure, or the like. In 1964 Muijdermann[(4)] and Booy[(5)] showed how, for incompressible lubricants in rectangular grooving, a correction solution can be found for the pressure. The addition of this solution to a Whipple-type solution produces a complete solution capable of satisfying a constant-pressure edge condition. Both of these authors obtained correction solutions for the case of isolated grooves (great disparity between groove and ridge film thicknesses), although Muijdermann offered for less extreme disparities a correction based largely on heuristic reasoning. Besides incorporating edge corrections, Muijdermann and Booy validated Whipple's treatment, as it applies to incompressible flow, by the matching of solutions of Laplace's equation. In addition, Muidjermann translated his straight parallel-groove results to spiral-groove geometry by conformal mapping, and adapted them to gaseous lubricants under conditions of no flow.

All continuum theories for grooved-surface bearings become incorrect when the molecular mean free-path is commensurate with film thickness. In 1968 Hsing and Malanoski[6] modified the Vohr-Pan theory to incorporate Maxwell's slip condition, and found deleterious effects on performance with the light gases, such as helium, neon and hydrogen.

Additional study of local compressibility effects (across grooving) was made for gas-lubricated herringbone bearings having rectangular grooving in 1968 by Constantinescu and Castelli.[7] Using detailed finite-difference solutions of the groove-ridge pressure distributions, and an analogy with step-slider-bearings, they concluded that Whipple's quasi-incompressible assumption is valid provided (our notation):

$$\epsilon \Lambda \leq (1.2-1.8)\ H_1^2; \qquad \text{(groove width = ridge width).}$$

They found the condition met in most current applications.

The foregoing references present to this writer's knowledge, all the concepts that have been published concerning basic theory for grooved surfaces in steady-state operation.

Many worthy design studies and experimental investigations have not been cited because they are not sufficiently relevant to the present work, which is concerned primarily with extending existing theory.

In the present work, a narrow-groove theory is developed by unified, consistent expansions in terms of the small parameter:

$$\epsilon \equiv \frac{\text{groove-ridge wavelength}}{\text{bearing dimension}} = \frac{\Delta}{L}$$

Just as in Whipple's original study, parallel-plate thrust-bearing geometry with straight, parallel grooving is assumed, except that the grooving may now possess arbitrary transverse section. An asymptotic expansion valid in the bearing interior (away from edges) yields Whipple's equation for the trend of the average pressure level, together with a second relation for the ripples of pressure induced by individual grooves and ridges. A second asymptotic expansion, valid only near the bearing edges, yields expressions for a generalized Muijdermann-Booy edge correction. When the two asymptotic expansions are matched, a theory demonstrating first-order effects of groove-ridge wavelength is obtained.

DEVELOPMENT OF DIFFERENTIAL EQUATIONS

Differential Equation in Skewed Coordinates

Figure 1 is a sketch of a section of the type of thrust plate to be analyzed. The opposing smooth plate (not shown) moves in the x-direction with velocity, U. Although rectangular grooves and pads are shown, the groove-land shapes may have any periodic dependence on "x". The film shape along the direction of the grooves is constant. All "corrugations" are straight and parallel.

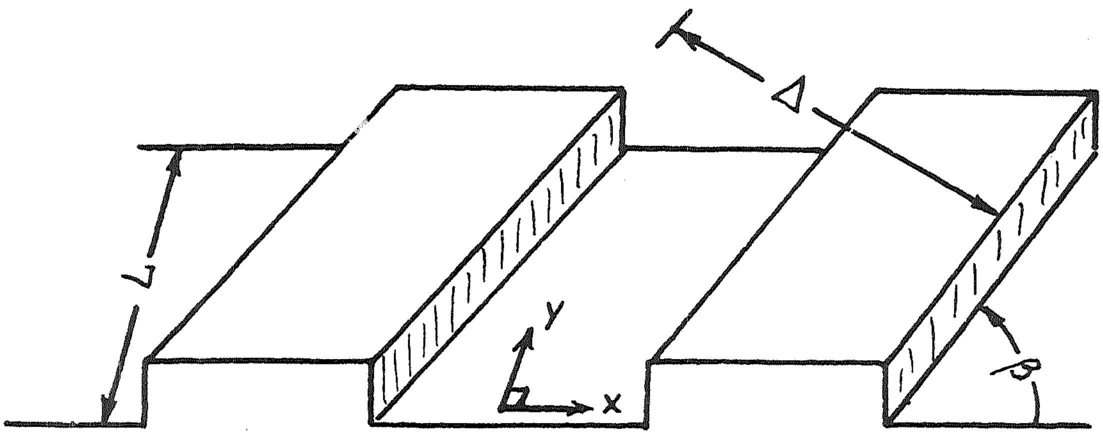


Figure 1 Schematic Diagram of Groove-Land Pairs

Figure 2 defines the skewed coordinate system in terms of which it is convenient to perform the analysis. Here:

$$z = (y - x\tan\beta)\cos\beta = y\cos\beta - x\sin\beta \quad [1]$$

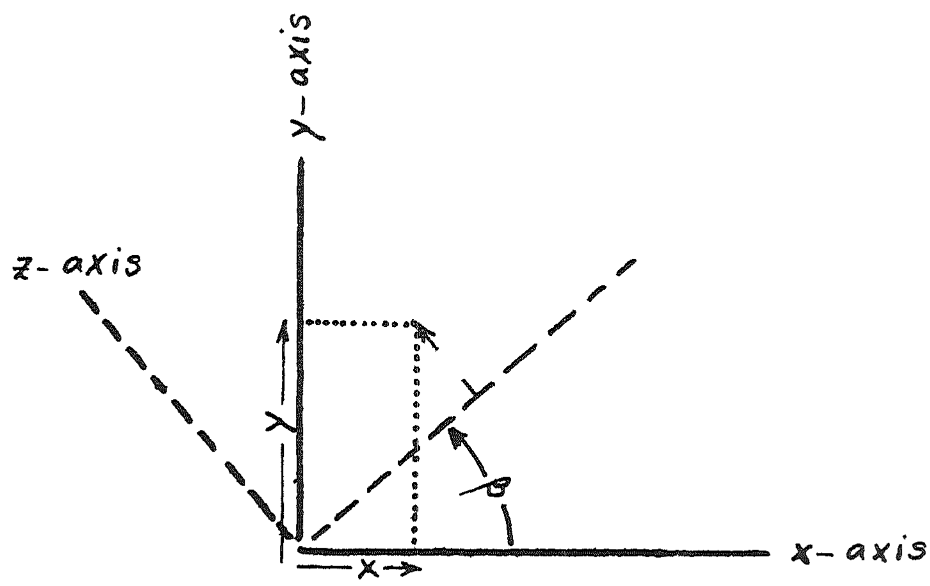


Figure 2 Skewed Coordinate System

Corresponding to eq. [1], the derivatives are:

$$\left.\frac{\partial}{\partial x}\right)_y = -\sin\beta \left.\frac{\partial}{\partial z}\right)_y \; ; \; \left.\frac{\partial}{\partial y}\right)_x = \left.\frac{\partial}{\partial y}\right)_z + \cos\beta \left.\frac{\partial}{\partial z}\right)_y \qquad [2]$$

Reynolds equation in Cartesian coordinates (x,y) is:

$$\frac{\partial}{\partial x}(pUh) = \frac{\partial}{\partial x}\frac{h^3}{6\mu}p\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\frac{h^3}{6\mu}p\frac{\partial p}{\partial y} \qquad [3]$$

Use of eqs. [2] converts eq. [3] to:

$$-\sin\beta\frac{\partial}{\partial z}(pUh) = \frac{\partial}{\partial z}\left(\frac{h^3}{6\mu}p\frac{\partial p}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{h^3}{6\mu}p\frac{\partial p}{\partial y}\right) + 2\cos\beta\frac{h^3}{6\mu}\frac{\partial}{\partial z}p\frac{\partial p}{\partial y} + \frac{\cos\beta}{6\mu}p\frac{\partial p}{\partial y}\frac{\partial h^3}{\partial z} \qquad [4]$$

Dimensionless variables are now introduced. Thus:

$$H \equiv \frac{h}{c}\ ; \quad \zeta \equiv \frac{z}{\Delta}\ ; \quad \xi \equiv \frac{y}{L}\ ; \quad \epsilon \equiv \frac{\Delta}{L}$$

$$\pi \equiv p/p_a\ ; \quad \Lambda \equiv \sin\beta\ \frac{6\mu U L}{p_a c^2} \qquad [5]$$

Note that ζ measures distance in terms of the groove-land wavelength, whereas ξ measures distance in terms of a principal overall bearing dimension, L.

In terms of the new variables, eq. [4] becomes:

$$-\epsilon\Lambda \frac{\partial}{\partial\zeta}(\pi H) = \frac{\partial}{\partial\zeta} H^3 \pi \frac{\partial\pi}{\partial\zeta} + \epsilon^2 H^3 \frac{\partial}{\partial\xi}\pi\frac{\partial\pi}{\partial\xi} + \epsilon\, 2(\cos\beta) H^3 \frac{\partial}{\partial\zeta}\pi\frac{\partial\pi}{\partial\xi} + \epsilon(\cos\beta)\,\pi\frac{\partial\pi}{\partial\xi}\frac{\partial H^3}{\partial\zeta} \qquad [6]$$

An alternative form to that of eq. [6] is convenient for certain operations. It is in terms of $\psi \equiv \pi^2$. Thus:

$$-\epsilon\, 2\Lambda \frac{\partial}{\partial\zeta}(H\sqrt{\psi}) = \frac{\partial}{\partial\zeta} H^3 \frac{\partial\psi}{\partial\zeta} + \epsilon^2 \frac{\partial}{\partial\xi} H^3 \frac{\partial\psi}{\partial\xi} + \epsilon(\cos\beta)\frac{\partial}{\partial\zeta} H^3 \frac{\partial\psi}{\partial\xi} + \epsilon(\cos\beta) H^3 \frac{\partial^2\psi}{\partial\zeta\,\partial\xi} \qquad [7]$$

Typically, values for $\pi(\zeta, 0)$ and $\pi(\zeta, 1)$ might be specified, and with $H(1+\zeta) = H(\zeta)$, periodic solutions for π and ψ would be sought.

Imposition of Periodicity

Integration of eq. [7] over one groove-land cycle (with H and ψ periodic) gives:

$$0 = \epsilon^2 \frac{d^2}{d\xi^2} \int_0^1 H^3 \psi \, d\zeta + \epsilon (\cos\beta) \frac{d}{d\xi} \int_0^1 H^3 \frac{\partial \psi}{\partial \zeta} d\zeta \qquad [8]$$

A second integration gives:

$$-\epsilon^2 \dot{\mathcal{M}} = \epsilon^2 \frac{d}{d\xi} \int_0^1 H^3 \psi \, d\zeta + \epsilon (\cos\beta) \int_0^1 H^3 \frac{\partial \psi}{\partial \zeta} d\zeta \qquad [9]$$

Here $\dot{\mathcal{M}}$ is the dimensionless mass "throughput" for a groove-land combination. This fact is verified by noting that the mass flux within the film which takes place in the y-direction, $\dot{m}_y$, is:

$$\dot{m}_y = -\frac{h^3}{12\mu}\left(\frac{p}{RT}\right) \left.\frac{\partial p}{\partial y}\right)_x = -\frac{h^3}{12\mu RT} p \left\{ \left.\frac{\partial p}{\partial y}\right)_z + \cos\beta \left.\frac{\partial p}{\partial z}\right)_y \right\} \qquad [10]$$

When put in dimensionless form, this equation becomes:

$$\frac{24\mu RT L \dot{m}_y}{c^3 p_a^2} = -H^3 \frac{\partial \psi}{\partial \xi} + \frac{\cos\beta}{\epsilon} H^3 \frac{\partial \psi}{\partial \zeta} \qquad [11]$$

Integration over a cycle identifies:

$$\dot{\mathcal{M}} = \frac{24\mu RT L}{c^3 p_a^2} \int_0^\Delta \dot{m}_y \, dz \qquad [12]$$

An indefinite integration of eq. [7] yields:

$$H^3\frac{\partial\psi}{\partial\zeta} = -\epsilon 2\Lambda H\sqrt{\psi} - \epsilon^2\frac{d^2}{d\xi^2}\int_0^\zeta H^3\psi\, d\zeta$$

$$-\epsilon(\cos\beta)H^3\frac{\partial\psi}{\partial\xi} - \epsilon(\cos\beta)\frac{d}{d\xi}\int_0^\zeta H^3\frac{\partial\psi}{\partial\zeta}d\zeta + f(\xi) \qquad [13]$$

For ψ to be periodic, it is necessary that:

$$\int_0^1\frac{\partial\psi}{\partial\zeta}d\zeta = 0 = -\epsilon 2\Lambda\int_0^1 H^{-2}\sqrt{\psi}\, d\zeta - \epsilon^2\frac{d^2}{d\xi^2}\int_0^1 H^{-3}\int_0^\zeta H^3\psi\, d\zeta\, d\zeta$$

$$-\epsilon(\cos\beta)\frac{d}{d\xi}\int_0^1\psi\, d\zeta - \epsilon(\cos\beta)\frac{d}{d\xi}\int_0^1 H^{-3}\int_0^\zeta H^3\frac{\partial\psi}{\partial\zeta}d\zeta\, d\zeta + \overline{H^{-3}}f(\xi) \qquad [14]$$

[The notation $\overline{H^n} \equiv \int_0^1 H^n d\zeta$ will be used for brevity.]

Now multiply eq. [13] by $\overline{H^{-3}}$, integrate over a cycle, and subtract eq. [14]. The function $f(\xi)$ is thereby eliminated, and the result is:

$$\overline{H^{-3}}\int_0^1 H^3\frac{\partial\psi}{\partial\zeta}d\zeta = \epsilon 2\Lambda\int_0^1\left(H^{-2} - H\overline{H^{-3}}\right)\sqrt{\psi}\, d\zeta + \epsilon^2\frac{d^2}{d\xi^2}\int_0^1\left(H^{-3} - \overline{H^{-3}}\right)\int_0^\zeta H^3\psi\, d\zeta\, d\zeta$$

$$+\epsilon(\cos\beta)\frac{d}{d\xi}\int_0^1\left(1 - H^3\overline{H^{-3}}\right)\psi\, d\zeta + \epsilon(\cos\beta)\frac{d}{d\xi}\int_0^1\left(H^{-3} - \overline{H^{-3}}\right)\int_0^\zeta H^3\frac{\partial\psi}{\partial\zeta}d\zeta\, d\zeta$$

This equation, in combination with eq. [9], yields an important condition which all ψ-solutions must satisfy; ie.,

$$
\begin{aligned}
-\dot{\mathcal{M}} = {} & \frac{d}{d\xi}\int_0^1 H^3\overline{H^{-3}}\psi\, d\zeta + \frac{\cos\beta}{\overline{H^{-3}}}\Bigg[\overline{H^{-3}}\int_0^1 2\Lambda\left(H^{-2} - H\overline{H^{-3}}\right)\sqrt{\psi}\, d\zeta \\
& + \epsilon\frac{d^2}{d\xi^2}\int_0^1 \left(H^{-3} - \overline{H^{-3}}\right)\int_0^\zeta H^3\overline{H^{-3}}\psi\, d\zeta\, d\zeta + (\cos\beta)\frac{d}{d\xi}\int_0^1 \left(1 - H^3\overline{H^{-3}}\right)\overline{H^{-3}}\psi\, d\zeta \\
& + (\cos\beta)\frac{d}{d\xi}\int_0^1 \left(H^{-3} - \overline{H^{-3}}\right)\int_0^\zeta \frac{\partial}{\partial\zeta}\left(\overline{H^{-3}}\psi\right)H^3\, d\zeta\, d\zeta\Bigg]
\end{aligned}
\qquad [16]
$$

ϵ-Expansion for Bearing Interior

To obtain specific results for ψ, a series expansion in "ϵ" is proposed. This series should become useful when the ratio of groove + land width, "Δ", to the overall bearing dimension, "L", is small. Thus:

$$\psi = \psi_0(\zeta,\xi) + \epsilon\,\psi_1(\zeta,\xi) + \epsilon^2\psi_2(\zeta,\xi) + \cdots \qquad [17]$$

When this series is inserted into eq. [7], and the coefficients of the various powers of "ϵ" are equated, a sequence of partial differential equations is obtained. The first two such equations are as follows:

$$\frac{\partial}{\partial\zeta}H^3\frac{\partial\psi_0}{\partial\zeta} = 0 \qquad [18]$$

$$-2\Lambda\sqrt{\psi_0}\,\frac{\partial H}{\partial \zeta} = \frac{\partial}{\partial \zeta}H^3\frac{\partial \psi_1}{\partial \zeta} + (\cos\beta)\frac{\partial}{\partial \zeta}\left(H^3\frac{\partial \psi_0}{\partial \xi}\right)$$

$$+ (\cos\beta)\,H^3\frac{\partial^2 \psi_0}{\partial \zeta\,\partial \xi} \qquad [19]$$

Integration of eq. [18] gives:

$$H^3\frac{\partial \psi_0}{\partial \zeta} = f(\xi) \qquad [20]$$

But in order for ψ_0 to be cyclic, $f(\xi) = 0$, and

$$\psi_0 = \psi_0(\xi) \qquad [21]$$

Equation [19] now reduces to:

$$-2\Lambda\sqrt{\psi_0}\,\frac{\partial H}{\partial \zeta} = \frac{\partial}{\partial \zeta}H^3\frac{\partial \psi_1}{\partial \zeta} + (\cos\beta)\,\frac{d\psi_0}{d\xi}\,\frac{dH^3}{d\zeta} \qquad [22]$$

After one integration, this becomes:

$$-2\Lambda\sqrt{\psi_0}\,H = H^3\frac{\partial \psi_1}{\partial \zeta} + (\cos\beta)\,\frac{d\psi_0}{d\xi}H^3 + f(\xi) \qquad [23]$$

Imposition of the cyclic condition on ψ_1 yields:

$$H^3\overline{H^{-3}}\,\frac{\partial \psi_1}{\partial \zeta} = (\cos\beta)\,\frac{d\psi_0}{d\xi}\left(1 - H^3\overline{H^{-3}}\right) + 2\Lambda\sqrt{\psi_0}\,\left(\overline{H^{-2}} - H\,\overline{H^{-3}}\right) \qquad [24]$$

Then:

$$\overline{H^{-3}}\,\psi_1 = (\cos\beta)\,\frac{d\psi_0}{d\xi}\int_0^{\zeta}\left(H^{-3} - \overline{H^{-3}}\right)d\zeta + 2\Lambda\sqrt{\psi_0}\int_0^{\zeta}\left(H^{-3}\overline{H^{-2}} - H^{-2}\overline{H^{-3}}\right)d\zeta$$

$$+ \alpha(\xi)\left(2\,\overline{H^{-3}}\sqrt{\psi_0}\right) \qquad [25]$$

The expressions for ψ_0 and ψ_1 both contain functions of ξ for which the governing equations must be found. Note that ψ_0 reflects the general level of p^2, whereas ψ_1 shows some groove-land ripple through its dependence on ζ.

<u>Zeroth-Order Interior Differential Equation</u>

The zeroth-order condition imposed by eq. [16] is:

$$-\dot{m}_0 = \frac{d\psi_0}{d\xi}\overline{H^3}\,\overline{H^{-3}} + 2(\cos\beta)\Lambda\left(\overline{H^{-2}} - \overline{H}\,\overline{H^{-3}}\right)\sqrt{\psi_0}$$

$$+(\cos^2\beta)\frac{d\psi_0}{d\xi}\left(1 - \overline{H^3}\,\overline{H^{-3}}\right) \qquad [26]$$

This ordinary differential equation is easily rearranged to assume the form:

$$\frac{d\psi_0}{d\xi} = 2\Lambda_{eff}\sqrt{\psi_0} - \frac{\dot{m}_0}{\left\{\cos^2\beta + \sin^2\beta\,\overline{H^3}\,\overline{H^{-3}}\right\}} \qquad [27]$$

where:

$$\Lambda_{eff} = \Lambda\frac{(\cos\beta)\left(\overline{H}\,\overline{H^{-3}} - \overline{H^{-2}}\right)}{\left\{\cos^2\beta + (\sin^2\beta)\,\overline{H^3}\,\overline{H^{-3}}\right\}} \qquad [28]$$

The terminal values of ψ_0 at $\xi = 0$ and $\xi = 1$ can be specified, or various combinations with the flow, $\dot{m}_0$.

First-Order Interior Differential Equation

The first-order condition imposed by eq. [16] is:

$$
\begin{aligned}
-\dot{\mathcal{M}}_1 = {} & \Lambda \frac{d\sqrt{\Psi_0}}{d\xi} K_1 + (\cos\beta)\frac{d^2\Psi_0}{d\xi^2} K_2 + \left\{\frac{d}{d\xi}\left(\alpha(\xi)\sqrt{\Psi_0}\right)\right\} 2\,\overline{H^{-3}}\;\overline{H^{3}} \\
& + \frac{(\cos\beta)}{\overline{H^{-3}}}\Bigg[\frac{\Lambda}{\sqrt{\Psi_0}}\left\{\Lambda\sqrt{\Psi_0}\,K_3 + (\cos\beta)\frac{d\Psi_0}{d\xi}K_4 + \alpha(\xi)\sqrt{\Psi_0}\,2\,\overline{H^{-3}}\left(\overline{H^{-2}} - \overline{H}\;\overline{H^{-3}}\right)\right\} \\
& + \frac{d^2\Psi_0}{d\xi^2}K_5 + (\cos\beta)\left\{\Lambda\frac{d\sqrt{\Psi_0}}{d\xi}K_6 + (\cos\beta)\frac{d^2\Psi_0}{d\xi^2}K_7 + \frac{d}{d\xi}\,\alpha(\xi)\sqrt{\Psi_0}\;2\,\overline{H^{-3}}\,K_8\right\} \\
& + (\cos\beta)\Lambda\frac{d\sqrt{\Psi_0}}{d\xi}K_9 + (\cos^2\beta)\frac{d^2\Psi_0}{d\xi^2}K_{10}\Bigg]
\end{aligned}
\qquad [29]
$$

Here the various K's are constants dependent on the film-thickness distribution, $H(\xi)$, only; i.e., independent of β. They are listed in Appendix A.

Rearrangement gives: (with $\pi_0 = \sqrt{\Psi_0}$)

$$
\begin{aligned}
-\dot{\mathcal{M}}_1 = {} & \left\{2\,\overline{H^{-3}}\;\overline{H^{3}} + (\cos^2\beta)\,2K_8\right\}\frac{d}{d\xi}\left\{\pi_0\,\alpha(\xi)\right\} \\
& + \frac{d^2\Psi_0}{d\xi^2}\left[(\cos\beta)K_2 + (\cos\beta)\frac{K_5}{\overline{H^{-3}}} + (\cos^3\beta)\frac{K_7}{\overline{H^{-3}}} + (\cos^3\beta)\frac{K_{10}}{\overline{H^{-3}}}\right] \\
& + 2\Lambda(\cos\beta)\left(\overline{H^{-2}} - \overline{H}\;\overline{H^{-3}}\right)\alpha(\xi) \\
& + \Lambda\frac{d\pi_0}{d\xi}\left[K_1 + \frac{(\cos^2\beta)K_6}{\overline{H^{-3}}} + 2\frac{(\cos^2\beta)K_4}{\overline{H^{-3}}} + \frac{(\cos^2\beta)K_9}{\overline{H^{-3}}}\right] \\
& + \Lambda^2\frac{(\cos\beta)K_3}{\overline{H^{-3}}}
\end{aligned}
\qquad [30]
$$

In this form, the relation for $\alpha(\xi)$ is clearly a first-order inhomogeneous linear differential equation with integration constant, $\dot{\mathcal{M}}_1$ (and hence is equivalent to a second-order equation). Boundary conditions on $\alpha(\xi)$ are to be determined through solution of an "edge" problem.

Reduction of the coefficients in eq. [30] is carried out in Appendix A. The result is:

$$\frac{d}{d\xi}(\pi_0\alpha) = \Lambda_{eff}\,\alpha - \frac{\Lambda}{\overline{H^{-3}}}\frac{d\pi_0}{d\xi}\left[\int_0^1 \frac{QF\,d\zeta}{2\overline{Q}} + \frac{\cot^2\beta}{\overline{Q}}\int_0^1 (H^{-3} - \overline{H^{-3}})\int_0^\zeta (\overline{H^{-2}} - H^{-2})\,d\zeta\,d\zeta\right]$$
$$+ \frac{\Lambda\Lambda_{eff}}{\overline{H^{-3}}}\int_0^1 \frac{RF\,d\zeta}{2\overline{R}} - \frac{\dot{\mathcal{M}}_1}{2\overline{Q}}$$

[31]

<u>The "Complete" Differential Equation</u>:

At this point it will be helpful to recapitulate. Recall that it was convenient to use in mathematical developments the function:

$$\psi = \pi^2$$
$$\therefore \psi_0 + \epsilon\psi_1 = \pi_0^2 + 2\epsilon\pi_0\pi_1$$

so that:

$$\sqrt{\psi_0} = \pi_0 \;;\; \psi_1 = 2\pi_0\pi_1 \qquad [32]$$

From eq. [21] it is seen that: $\Pi_0 = \Pi_0(\xi)$ and from eq. [27] it follows that:

$$\frac{d\Pi_0}{d\xi} = \Lambda_{eff} - \frac{\dot{M}_0}{2\Pi_0 \overline{Q}} \qquad [33]$$

Also, from eqs. [32] and [25], it is readily found that:

$$\Pi_1 = \frac{\Lambda}{\overline{H^{-3}}} F(\zeta) + \frac{(\overline{\cos^2\beta})}{\overline{H^{-3}}} \frac{d\Pi_0}{d\xi} G(\zeta) + \alpha(\xi) \qquad [34]$$

The differential eq. [31] gives, through $\alpha(\xi)$, the manner in which the first-order correction to the general pressure level propagates across the bearing.

A "complete" differential equation, implying both [33] and [31] can be derived as follows. From eq. [34]:

$$\alpha(\xi) = \overline{\Pi_1} - \frac{\Lambda}{\overline{H^{-3}}} \overline{F} - \frac{(\overline{\cos^2\beta})}{\overline{H^{-3}}} \frac{d\Pi_0}{d\xi} \overline{G} \qquad [35]$$

Insertion in eq. [31] gives, after some cancellation and rearrangement:

$$\frac{d}{d\xi}(\Pi_0 \overline{\Pi_1}) = \Lambda_{eff} \overline{\Pi_1} - \Lambda \frac{d\Pi_0}{d\xi} C_1 + \frac{\Lambda \Lambda_{eff}}{\overline{H^{-3}}} C_2 - \frac{\dot{M}_1}{2\overline{Q}} \qquad [36]$$

where
$$C_1 = \frac{1}{\overline{H^{-3}}} \left\{ \frac{\int_0^1 QF\,d\zeta}{2\overline{Q}} - \overline{F} + \frac{\overline{\cos^2\beta}}{\overline{Q}} \int_0^1 (H^{-3} - \overline{H^{-3}}) \int_0^{\zeta} (\overline{H^{-2}} - H^{-2})\, d\zeta\, d\zeta \right\} \qquad [37]$$

$$C_2 \equiv \frac{\int_0^1 RF\,d\zeta}{2\overline{R}} - \overline{F} \qquad [38]$$

Now multiply eq. [36] by "ϵ" and add to eq. [33] multiplied by $\overline{\pi}_o$. Thus:

$$\frac{d}{d\xi}\left(\frac{\overline{\pi}_o^2}{2} + \epsilon\,\overline{\pi}_o\overline{\pi}_1\right) = \Lambda_{eff}\left(\overline{\pi}_o + \epsilon\,\overline{\pi}_1\right) - \left(\dot{\mathcal{M}}_o + \epsilon\,\dot{\mathcal{M}}_1\right)$$

$$- \epsilon\,\Lambda\, C_1 \frac{d\overline{\pi}_o}{d\xi} + \epsilon\,\frac{\Lambda\Lambda_{eff}}{\overline{H^{-3}}}\, C_2 \qquad [39]$$

Through $O(\epsilon)$, $\quad \overline{\pi} = \overline{\pi}_o + \epsilon\,\overline{\pi}_1$

$$\overline{\pi^2} = \overline{\pi}_o^2 + \epsilon\, 2\overline{\pi}_o\overline{\pi}_1 = (\overline{\pi})^2 \qquad [40]$$

$$\therefore \frac{d}{d\xi}\left(\frac{\overline{\pi}^2}{2}\right) = \Lambda_{eff}\,\overline{\pi} - \frac{\dot{\mathcal{M}}}{2Q} - \epsilon\,\Lambda\, C_1 \frac{d\overline{\pi}}{d\xi} + \epsilon\,\frac{\Lambda\Lambda_{eff}}{\overline{H^{-3}}}\, C_2$$

[41]

Or:

$$(\overline{\pi} + \epsilon\Lambda C_1)\frac{d}{d\xi}(\overline{\pi} + \epsilon\Lambda C_1) = \Lambda_{eff}(\overline{\pi} + \epsilon\Lambda C_1)$$

$$- \frac{\dot{\mathcal{M}}}{2Q} + \epsilon\,\frac{\Lambda\Lambda_{eff}}{\overline{H^{-3}}} \qquad [42]$$

Finally, with

$$\dot{\mathcal{M}}^* \equiv \dot{\mathcal{M}} - \epsilon \frac{2\overline{Q}}{\overline{\overline{H^{-3}}}} \Lambda \Lambda_{eff} \left(C_2 - \overline{H^{-3}} C_1 \right) \qquad [43]$$

$$\pi^* \equiv \overline{\pi} + \epsilon \Lambda C_1 \qquad [44]$$

this equation reduces to:

$$\frac{d}{d\xi}\left(\frac{\pi^{*2}}{2}\right) = \Lambda_{eff}\, \pi^* - \frac{\dot{\mathcal{M}}^*}{2Q} \qquad [45]$$

Thus it is seen that a modified Whipple equation can accurately account for first-order groove-width effects.

Boundary conditions to which this differential equation is subject will be discussed subsequently.

DEVELOPMENT OF BOUNDARY CONDITIONS

Edge-Effect Expansion:

Now it should be noted that, at the entrance to the bearing, $\xi = 0$, the deviational pressure π_1 should be identically zero -- a condition that expression [34] manifestly cannot meet. Re-examination of the differential equation [7] and the expansion [17] is required. Use of the ϵ-expansion with dependent variables (ζ, ξ) implies that derivatives in terms of them are of equal order.

However, along $\xi = 0$ the pressure is constant, and $\frac{\partial \pi}{\partial \zeta} = 0$, $\qquad \frac{\partial}{\partial \xi} \neq 0$. Accordingly, new variables (ζ, η) are used with groove-ridge wavelength Δ as the measure of length in both the "z" and "y" directions; i.e.,

$$\eta \equiv \xi / \epsilon$$

Equation [6] now becomes:

$$-\epsilon \Lambda \frac{\partial}{\partial \zeta}(\pi H) = \frac{\partial}{\partial \zeta} H^3 \pi \frac{\partial \pi}{\partial \zeta} + H^3 \frac{\partial}{\partial \eta} \pi \frac{\partial \pi}{\partial \eta}$$

$$+ 2 (\cos\beta) H^3 \frac{\partial}{\partial \zeta} \pi \frac{\partial \pi}{\partial \eta} + (\cos\beta) \pi \frac{\partial \pi}{\partial \eta} \frac{\partial H^3}{\partial \zeta} \qquad [46]$$

The following expansion is now adopted. Thus:

$$\pi = \pi^{(0)}(\zeta, \eta) + \epsilon \pi^{(1)}(\zeta, \eta) + \cdots \qquad [47]$$

To effect asymptotic matching between this expansion and the earlier one, one writes:

$$\pi^{(0)}(\zeta,\eta) + \epsilon\, \pi^{(1)}(\zeta,\eta) \sim \tilde{\pi}_0(\zeta,\xi) + \epsilon\, \tilde{\pi}_1(\zeta,\xi) + \cdots$$

$$\sim \tilde{\pi}_0(\zeta,\epsilon\eta) + \epsilon\, \tilde{\pi}_1(\zeta,\epsilon\eta) + \cdots$$

$$\sim \tilde{\pi}_0(\zeta,0) + \epsilon\left\{\tilde{\pi}_1(\zeta,0) + \eta\,\frac{\partial\tilde{\pi}_0}{\partial\xi}\right\} + \cdots \qquad [48]$$

As $\eta \to \infty$, it is necessary that:

$$\pi^{(0)}(\zeta,\eta) \to \tilde{\pi}_0(\zeta,0) = 1 \qquad [49]$$

and

$$\pi^{(1)}(\zeta,\eta) \to \tilde{\pi}_1(\zeta,0) + \eta\left.\frac{\partial\tilde{\pi}_0}{\partial\xi}\right)_0 \qquad [50]$$

Now the substitution of series [47] into [46] gives for $(\pi^{(0)})^2$ a linear equation. Since $\pi^{(0)}(\zeta,0) = 1$ and $\pi^{(0)}(\zeta,0) = 1$, an obvious solution is: $\pi^{(0)} = 1$.

The differential equation for $\pi^{(1)}$ is:

$$-\Lambda\frac{\partial H}{\partial\zeta} = \frac{\partial}{\partial\zeta}H^3\frac{\partial\pi^{(1)}}{\partial\zeta} + \frac{\partial}{\partial\eta}H^3\frac{\partial\pi^{(1)}}{\partial\eta} + 2(\cos\beta)\,H^3\frac{\partial^2\pi^{(1)}}{\partial\zeta\,\partial\eta}$$

$$+ (\cos\beta)\,\frac{\partial\pi^{(1)}}{\partial\eta}\frac{\partial H^3}{\partial\zeta} \qquad [51]$$

This is the differential equation for operation with an incompressible fluid, and use of a correction for edge effect similar to that proposed by Muijdermann can be anticipated.

It is helpful to introduce a residual deviation,

$$\pi^{(1)} \equiv q + \pi_1(\zeta, 0) + \eta \left.\frac{\partial \pi_0}{\partial \xi}\right)_0 \qquad [52]$$

The function q is found to satisfy the following differential equation (by virtue of eq. [23]):

$$0 = \frac{\partial}{\partial \zeta} H^3 \frac{\partial q}{\partial \zeta} + \frac{\partial}{\partial \eta} H^3 \frac{\partial q}{\partial \eta} + 2(\cos\beta) H^3 \frac{\partial^2 q}{\partial \zeta \partial \eta} + (\cos\beta) \frac{\partial q}{\partial \eta} \frac{\partial H^3}{\partial \zeta} \qquad [53]$$

The appropriate boundary conditions are:

$$q(\zeta, 0) = -\pi_1(\zeta, 0)$$

$$= -\frac{\Lambda}{\overline{H^{-3}}} F(\zeta) - \frac{(\cos\beta)}{\overline{H^{-3}}} \frac{d\pi_0}{d\xi} G(\zeta) - \alpha(0) \qquad [54]$$

and:

$$q(\zeta, \eta) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty . \qquad [55]$$

Also, of course q is periodic in ζ .

<u>Boundary Conditions for the Pressure Correction</u>

It will be observed that $\alpha(0)$ is unknown. However, it can assume only one value compatible with $q \rightarrow 0$ as $\eta \rightarrow \infty$. It is the determination of this unique value which provides the entrance boundary condition for $\alpha(0)$. At the exit, $\zeta = 1$, similar analysis provides $\alpha(1)$. The differential equation

[31] connects these two values of "α". For chosen "β" and $H(\zeta)$, $\alpha(0)$ always depends linearly on "Λ" and "$\frac{d\Pi_0}{d\xi}$". However, for constant groove and land film thicknesses, the appropriate expressions are:

$$F(\zeta) = 2\left(H_1^{-3}\overline{H^{-2}} - \overline{H^{-3}}H_1^{-2}\right) \mathcal{S}t(\zeta, r)$$
$$G(\zeta) = \left(H_1^{-3} - \overline{H^{-3}}\right) \mathcal{S}t(\zeta, r) \qquad [56]$$

where $\mathcal{S}t(\zeta, r)$ is the sawtooth function defined as:

$$\mathcal{S}t(\zeta, r) = \zeta \quad \text{for} \quad 0 \leq \zeta \leq r$$
$$= (1-\zeta)\frac{r}{1-r} \quad \text{for} \quad r \leq \zeta \leq 1 \qquad [57]$$

$$\therefore \; q(\zeta, 0) = -\frac{1}{\overline{H^{-3}}}\left\{2\Lambda\left(H_1^{-3}\overline{H^{-2}} - \overline{H^{-3}}H_1^{-2}\right) + (\cos\beta)\left(\frac{d\Pi_0}{d\xi}\right)_0\left(H_1^{-3} - \overline{H^{-3}}\right)\right\} \mathcal{S}t(\zeta, 0) - \alpha(0) \qquad [58]$$

In this case it is clear that the effects of speed and flow ($\frac{d\Pi_0}{d\xi}$ is influenced by flow) coalesce.

<u>Exact Solution for Muijdermann's First Correction</u>

The edge problem cannot ordinarily be solved in closed form. However, for the special, but very important, case of constant-depth grooves and constant-height ridges some progress can be made. Consider first the case of the groove film-thickness greatly exceeding the ridge film thickness. (First treated by Muijdermann with the use of an analog computer). In this case, eq. [53] reduces to Laplaces equation in the skewed coordinates. Thus:

$$\frac{\partial^2 q}{\partial \zeta^2} + \frac{\partial^2 q}{\partial \eta^2} + 2(\cos\beta)\frac{\partial^2 q}{\partial \zeta \partial \eta} = 0 \qquad [59]$$

with $\frac{\partial q}{\partial \eta} = \frac{\partial}{\partial \zeta} + (\cos\beta)\frac{\partial q}{\partial \eta} = 0$ at $\zeta = 0$ and $\zeta = 1$, say,

and $q(\zeta, 0) = -\{1 - \zeta + \alpha(0)\}$; $q(\zeta, \infty) = 0$

Here recognition of the linearity of the problem has lead to the removal of constants of amplitude.

An entirely equivalent problem is to seek a solution for $\tilde{q} - \tilde{q}_\infty = -q$, such that $\tilde{q}(\zeta, 0)$ cancels the ripple functions $\frac{\Lambda}{\overline{H^{-3}}} F(\zeta) + \frac{(\cos\beta)}{\overline{H^{-3}}} \frac{d\pi_0}{d\chi} G(\zeta)$ of $\pi_1(\zeta, 0)$. Here, then, $\tilde{q}(\zeta, 0) = 1 - \zeta$ and $\alpha(0) = -\tilde{q}_\infty$

As Muijdermann has pointed out, the asymptotic value of $\tilde{q}$ need not be the average of q along the base $\eta = 0$. However, this result <u>is</u> true for $\beta = \frac{\pi}{2}$. Thus:

$$\int_0^1 \frac{\partial^2 \tilde{q}}{\partial \zeta^2} d\zeta + \frac{d^2}{d\eta^2} \int_0^1 \tilde{q}\, d\zeta = 0 \qquad [60]$$

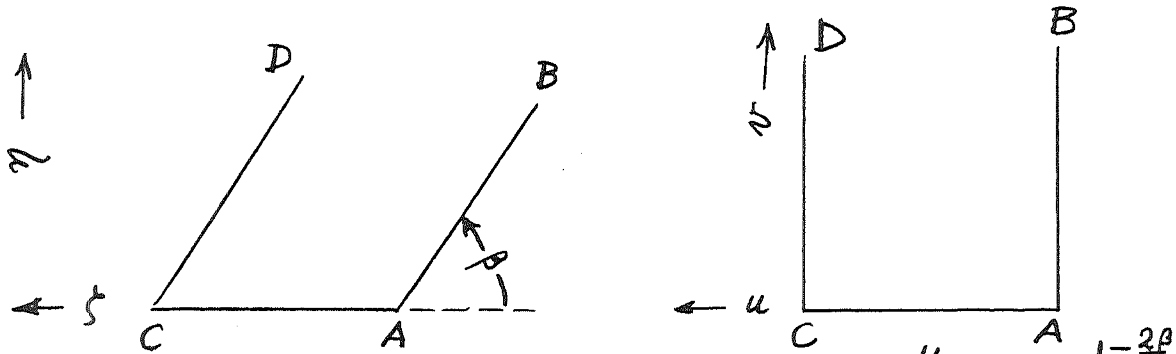
But $$\int_0^1 \frac{\partial^2 \tilde{q}}{\partial \zeta^2}\, d\zeta = \frac{\partial q}{\partial \zeta}\Big)_1 - \frac{\partial q}{\partial \zeta}\Big)_0 = 0 \qquad [61]$$

$$\therefore \int_0^1 \tilde{q}\, d\zeta = A + B\eta \qquad [62]$$

But if $\tilde{q} \to \tilde{q}_\infty$, B = 0 necessarily

$$\therefore \tilde{q}_\infty = \int_0^1 \tilde{q}(\zeta, 0)\, d\zeta \qquad [63]$$

The foregoing fact is utilized to obtain a closed-form solution for $\tilde{q}_\infty$ for the system [59]. By conformal mapping, problem A below is transformed to problem B. Now



$\tilde{q}(\zeta, 0) = (1-\zeta)$ $\qquad$ $\tilde{q}(u,0) = \sin\beta \int_0^u \left\{\cot\left(\frac{\pi u}{2}\right)\right\}^{1-\frac{2\beta}{\pi}} du$

Problem A. $\qquad$ Problem B.

Figure 3 <u>Conformal Transformation</u>

the average value of $\tilde{q}(\zeta, 0)$ in the (u, ν) plane (problem B) is also $\tilde{q}_\infty$ in the (u, ν) plane. However, <u>both planes possess the same</u> $\tilde{q}_\infty$. The determination of $\tilde{q}_\infty$ is, then, reduced to a definite integration along $\nu = 0$. Details are given in Appendix B. The final result is:

$$\hat{q}_\infty = 1 - \frac{\tan\beta}{\pi}\left(\psi\left(1-\frac{\beta}{\pi}\right) - \psi\left(\tfrac{1}{2}\right)\right) \qquad [64]$$

where ψ is the Digamma function.

For his "Δp_{corr}", Muijdermann plotted $2(\hat{q}_\infty - \frac{1}{2})$. Figure 4 compares his results with eq. [64]. Discrepancies are, for the most part, slight.

It is easy to see that the "response" as $\zeta \to 0$ to $q(\zeta, 0) = -\zeta$ is simply $-\frac{\tan\beta}{\pi}\left(\psi\left(1-\frac{\beta}{\pi}\right) - \psi\left(\frac{1}{2}\right)\right)$ and, therefore;

$$\hat{q}_\infty = \frac{\tan\beta}{\pi}\left(\psi\left(1-\frac{\beta}{\pi}\right) - \psi\left(\tfrac{1}{2}\right)\right) \qquad [65]$$

if $q(\zeta, 0) = \zeta$

Numerical Results for Finite Ridge Film Thickness, Rectangular Grooves

Except for the case of small film-thickness excursions ($H_1 \cong H_2$), the case of finite ridge-film-thickness has not proven to be analytically tractable. However, the finite-element numerical approach developed at the Franklin Institute[8] has been used to obtain numerical results. For the data presented in Figure 4 (with $H_1/H_2 \neq 0$) the author is much indebted to Dr. T.Y. Chu.

with boundary conditions of the form [144] applied locally along the exposed film edges.

In summation, it may be said that a narrow-groove theory has been developed here for parallel straight grooving of arbitrary transverse shape with aligned bearing surfaces. The consequences of the theory are generalizations of Whipple's analysis for the interior pressure distribution and of the Muijdermann-Booy analysis for edge effects. The expansion adopted in this work shows that, for sufficiently narrow grooves, the variations of density of a gas can be neglected in distances of the order of one groove width. The expansion is also valid for non-cavitating liquids --- one simply substitutes $\beta_T \equiv (\partial p/\partial \ln \rho)_T$ for "p_a" in all expressions, and retains all terms of numerical consequence.[9]

There are a number of ways in which the present analysis can be exploited and extended. In particular, future effort will be devoted to the development of a general partial differential equation corresponding to eq. [142], and to the provision of additional edge-correction information, of which that depicted in Figs. 4 and 5 is to be regarded simply as representative.

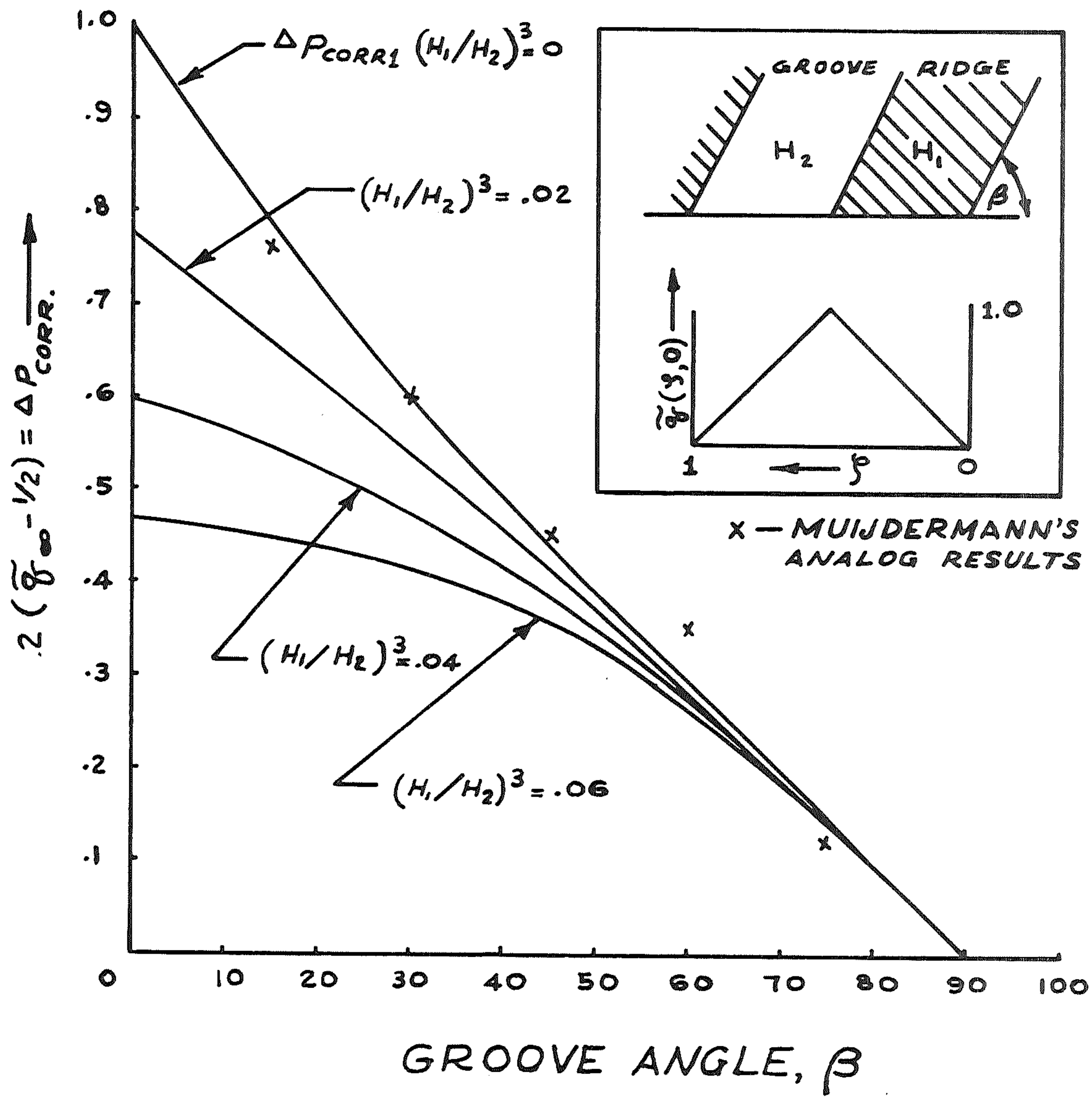


Fig. 4. Pressure Correction Factors for Edge Effect with Rectangular Grooving (Versus Groove Angle)

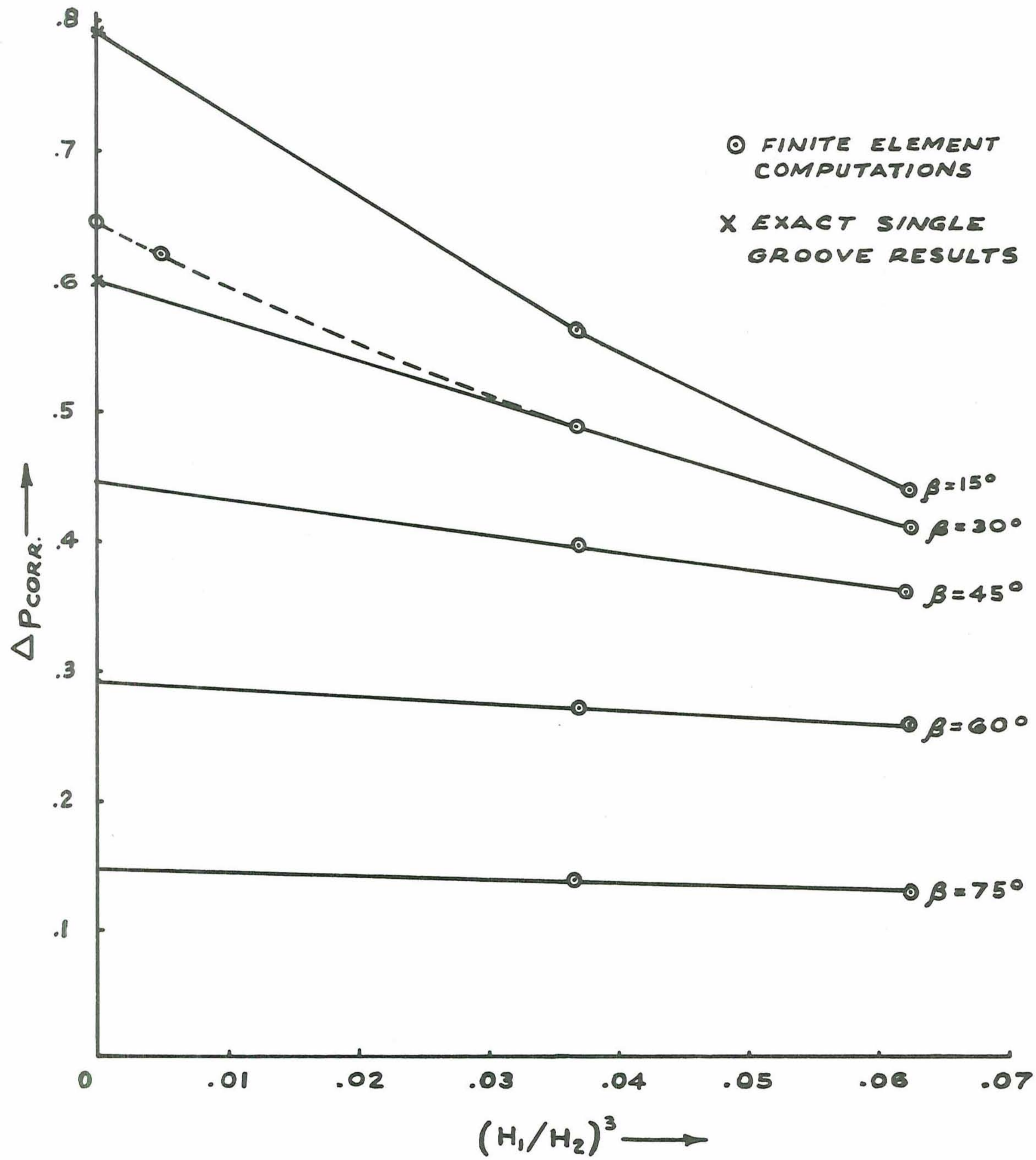


Fig. 5. Pressure Correction Factors for Edge Effect with Rectangular Grooving (Versus Film-Thickness Ratio)

If the author interprets Dr. Muijdermann's "second correction" expression properly, it would give:

$$\Delta p_{Corr} = \left(\Delta p_{Corr1}\right) \left[\frac{1-\left(\frac{H_1}{H_2}\right)^3}{1+\left(\frac{H_1}{H_2}\right)^3}\right] \qquad [66]$$

Although for $\beta \neq 0$ this formula gives correct limiting values, it would appear to be considerably in error for intermediate values (i.e., for $H_1/H_2 \neq 0$ and $\neq 1$). For example, with $\beta = 15^{\circ}$ and $(H_1/H_2)^3 = .06$ Figure 4 gives $\Delta p_{Corr} = 0.43$, whereas eq. [66] gives:

$$\Delta p_{Corr} = 0.79 \left[\frac{1-0.06}{1+0.06}\right]$$

The discrepancy in Δp_{Corr} is substantial. However, we must recall that we are concerned here with corrections to the pressures and load capacities, rather than the principal quantities themselves. Hence discrepancies such as that above will not introduce equally serious errors in overall predictions of performance.

To corroborate the finite-element results, the data were cross-plotted as functions of $(H_1/H_2)^3$. Only this parameter and the groove angle, β, appear in the rectangular groove (and ridge) problem. A perturbation treatment in $(H_1/H_2)^3$ would lead to a linear behavior for small values of this parameter. Figure 5 shows that the computed data do satisfy this criterion fairly well. Furthermore, in the case of $\beta = 30^\circ$, additional points near zero are shown to be tolerably in accord with the exact result as $(H_1/H_2)^3 \to 0$.

Perturbation Analysis with Finite Ridge-Film Thickness, Arbitrary Groove Shape

In the case of small excursions of film thickness, it is possible to make a complete analysis. For this analysis, the earlier results of Wildmann for a cosinusoidally-rippled herringbone are highly suggestive. Suppose that:

$$H = 1 + \delta\, f(\zeta) \qquad [67]$$

where δ is a small, amplitude parameter and $f(\zeta)$ is arbitrary except that:

$f(\zeta) = f(1 + \zeta)$ and $\overline{f(\zeta)} = 0$. The datum for ζ is chosen so that $f(0) = 0$.

It is readily shown that:

$$\left.\begin{array}{ll} H = 1 + \delta f & \overline{H} = 1 \\ H^{-1} = 1 - \delta f + \delta^2 f^2 + \cdots & \overline{H^{-1}} = 1 + \delta^2 \overline{f^2} \\ H^{-2} = 1 - 2\delta f + 3\delta^2 f^2 + \cdots & \overline{H^{-2}} = 1 + 3\delta^2 \overline{f^2} \\ H^{-3} = 1 - 3\delta f + 6\delta^2 f^2 + \cdots & \overline{H^{-3}} = 1 + 6\delta^2 \overline{f^2} \end{array}\right\} \qquad [68]$$

Also: $$F(\zeta) = 2\int_0^{\zeta}\{-\delta f + 3\,\delta^2(f^2-\overline{f^2})\}\,d\zeta + \cdots \qquad [69]$$

$$G(\zeta) = \int_0^{\zeta}\{-3\delta f + 6\,\delta^2(f^2-\overline{f^2})\}\,d\zeta + \cdots \qquad [70]$$

$$\Lambda_{eff} = \delta^2(3\Lambda\cos\beta) + \cdots \qquad [71]$$

Reference to eq. [33] shows that, in a no-flow situation, $\pi_0 - 1 = O(\delta^2)$. Let us confine consideration to cases where the flow through the bearing is of no greater magnitude (thereby limiting the degree of external pressurization). From eq. [34] we then note that:

$$\begin{aligned}\pi_1 &= \Lambda F(\zeta) + \alpha(\xi) + O(\delta^3)\\ &= -2\Lambda\delta\int_0^{\zeta} f\,d\zeta + 6\Lambda\delta^2\int_0^{\zeta}(f^2-\overline{f^2})\,d\zeta + \alpha(\xi) + O(\delta^3)\end{aligned}$$

[72]

the residual function q must cancel $\pi_1(\zeta,0)$ and satisfy the differential eq. [53]:

$$\frac{\partial^2 q}{\partial\zeta^2} + \frac{\partial^2 q}{\partial\eta^2} + 2(\cos\beta)\frac{\partial^2 q}{\partial\zeta\partial\eta} + \frac{\partial \ln H^3}{\partial\zeta}\left(\frac{\partial q}{\partial\zeta} + \cos\beta\,\frac{\partial q}{\partial\eta}\right) = 0$$

[73]

In the present case,

$$\frac{\partial \ln H^3}{\partial\zeta} = 3\delta f'(\zeta) + O(\delta^3) \qquad [74]$$

Now adopt the expansion:

$$q = \delta\, q_1(\xi,\eta) + \delta^2 q_2(\xi,\eta) + \text{ etc.} \qquad [75]$$

the differential equations for q_1 and q_2 are:

$$\frac{\partial^2 q_1}{\partial \xi^2} + \frac{\partial^2 q_1}{\partial \eta^2} + 2(\cos\beta)\frac{\partial^2 q_1}{\partial \xi \partial \eta} = 0 \qquad [76]$$

and:

$$\frac{\partial^2 q_2}{\partial \xi^2} + \frac{\partial^2 q_2}{\partial \eta^2} + 2(\cos\beta)\frac{\partial^2 q_2}{\partial \xi \partial \eta} + 3f'\left(\frac{\partial q_1}{\partial \xi} + \cos\beta \frac{\partial q_1}{\partial \eta}\right) = 0 \qquad [77]$$

On the supposition that q_1 can be represented in the form

$$q_1 = Q(\eta)\, e^{2\pi i k \xi} \qquad [78]$$

we find:

$$Q''(\eta) + (2\pi i k)(2\cos\beta)\, Q'(\eta) - (2\pi k)^2 Q(\eta) = 0 \qquad [79]$$

Or: $$q_1 = (\text{const})\; e^{\pm \eta\, 2\pi k \sin\beta}\; e^{2\pi k i(\xi - \eta\cos\beta)} \qquad [80]$$

More generally, let us take:

$$\tilde{q}_1 = \sum_{k=-\infty}^{\infty} A_k\, e^{-|k|(2\pi \sin\beta)\eta}\; e^{2\pi i k(\xi - \eta\cos\beta)} \qquad [81]$$

At $\eta = 0$, temporarily let:

$$\mathcal{F}(\zeta) \equiv \int_0^{\zeta} f\,d\zeta = \tilde{q}(\zeta, 0) = \sum_{k=-\infty}^{\infty} A_k\, e^{2\pi i k \zeta} \qquad [82]$$

The differential equation for $\tilde{q}_2$ is, with the use of eq. [68]

$$\frac{\partial^2 \tilde{q}_2}{\partial \zeta^2} + \frac{\partial^2 \tilde{q}_2}{\partial \eta^2} + 2(\cos\beta)\frac{\partial^2 \tilde{q}_2}{\partial\zeta\,\partial\eta} - 3(\sin\beta)\, f'(\zeta) \sum_{k=-\infty}^{\infty} 2\pi|k|\, A_k\, e^{-\psi_k}\, e^{2\pi i k\zeta} = 0 \qquad [83]$$

where: $$\psi_k = 2\pi k \eta \left\{ i\cos\beta + \mathrm{sgn}(k)\sin\beta \right\} + i\,\mathrm{sgn}(k)\beta \qquad [84]$$

Integration of eq. [83] over a groove-ridge cycle gives:

$$\frac{d^2 \bar{q}_2}{d\eta^2} = 3(\sin\beta) \sum_{k=-\infty}^{\infty} 2\pi|k|\, A_k\, e^{-\psi_k} \int_0^1 f'(\zeta)\, e^{2\pi i k\zeta}\, d\zeta \qquad [85]$$

But: $$f(\zeta) = \sum_{-\infty}^{\infty} 2\pi i k\, A_k\, e^{2\pi i k\zeta}; \quad f'(\zeta) = -\sum_{-\infty}^{\infty} (2\pi k)^2 A_k\, e^{2\pi i k\zeta} \qquad [86]$$

So that:

$$\frac{d^2 \bar{q}_2}{d\eta^2} = -3(\sin\beta) \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} 2\pi|k|\, A_k\, e^{-\psi_k} (2\pi j)^2 A_j \int_0^1 e^{2\pi i\zeta(k+j)}\, d\zeta \qquad [87]$$

$$= -3(\sin\beta) \sum_{j=-\infty}^{\infty} (2\pi j)^3 (\mathrm{sgn}\, j)\, A_j A_{-j}\, e^{-\psi_j} \qquad [88]$$

$$= -3(\sin\beta) \sum_{j=-\infty}^{\infty} (2\pi j)^3 (\mathrm{sgn}\, j)\, A_j A_{-j}\, e^{-i(\mathrm{sgn}\, j)\beta}\, e^{-2\pi j \eta i\, e^{-i(\mathrm{sgn}\, j)\beta}} \qquad [89]$$

Note that all terms vanish as $\eta \to \infty$.

Equation [89] is now integrated twice, with the linear term in "y" set equal to zero. Then, taking account of the temporarily omitted "lead constant" on $\tilde{q}_1$, one gets:

$$\overline{\tilde{q}_2}(0) = -6\Lambda(\sin\beta)\sum_{j=-\infty}^{\infty} 2\pi|j| A_j A_{-j}\, e^{i(\operatorname{sgn} j)\beta} + \overline{\tilde{q}_2}(\infty) \qquad [90]$$

$$= -6\Lambda(\sin\beta)\left[\sum_{1}^{\infty} 2\pi k\, A_k A_{-k}\, e^{i\beta} + \sum_{1}^{\infty} 2\pi k\, A_k A_{-k}\, e^{-i\beta}\right] + \overline{\tilde{q}_2}(\infty) \qquad [91]$$

Or:

$$\overline{\tilde{q}_2}(\infty) = \overline{\tilde{q}_2}(0) + 6\Lambda(\sin 2\beta)\sum_{k=1}^{\infty} 2\pi k\, A_k A_{-k} \qquad [92]$$

On $\eta = 0$:

$$\delta\,\tilde{q}_1(\xi,0) + \delta^2\tilde{q}_2(\xi,0) = -2\Lambda\delta\mathcal{F} + 6\Lambda\delta^2\mathcal{S} \qquad [93]$$

where

$$\mathcal{S}(\xi) \equiv \int_0^{\xi} (f^2 - \overline{f^2})\, d\xi \qquad [94]$$

Then:

$$\left.\begin{aligned} \overline{\tilde{q}_1}(0) &= -2\Lambda\overline{\mathcal{F}} = \overline{\tilde{q}_1}(\infty) \\ \overline{\tilde{q}_2}(0) &= 6\Lambda\overline{\mathcal{S}} \end{aligned}\right\} \qquad [95]$$

And:

$$\overline{\tilde{q}}(\infty) = -\alpha(0) = \delta\,\overline{\tilde{q}_1}(0) + \delta^2\left\{\overline{\tilde{q}_2}(0) + 6\Lambda\sin 2\beta\sum_{k=1}^{\infty} 2\pi k\, A_k A_{-k}\right\} \qquad [96]$$

$$= \overline{\tilde{q}}(0) + 6\Lambda\delta^2(\sin 2\beta)\sum_{k=1}^{\infty} 2\pi k\, A_k A_{-k} \qquad [97]$$

Further analysis obviates the need for Fourier decomposition. Since

$$\mathcal{F}(\zeta) = \sum_{k=-\infty}^{\infty} A_k e^{2\pi i k \zeta} \qquad [98]$$

$$A_k = \int_0^1 \mathcal{F}(\zeta) e^{-2\pi i k \zeta} d\zeta \qquad [99]$$

$$\therefore A_k A_{-k} = \int_0^1 \mathcal{F}(s) e^{-2\pi i k s} ds \int_0^1 \mathcal{F}(t) e^{2\pi i k t} dt \qquad [100]$$

Or:

$$A_k A_{-k} = \int_0^1 \int_0^1 \mathcal{F}(s) \mathcal{F}(t) e^{2\pi i k (t-s)} ds\, dt \qquad [101]$$

$$= \int_0^1 \int_0^1 \mathcal{F}(s) \mathcal{F}(t) e^{-2\pi i k (t-s)} ds\, dt \qquad [102]$$

$$= \int_0^1 \int_0^1 \mathcal{F}(s) \mathcal{F}(t) \cos 2\pi k (s-t)\, ds\, dt \qquad [103]$$

$$= \int_0^1 \int_0^1 \mathcal{F}(s) \mathcal{F}(t) \left\{ \cos 2\pi k s \cos 2\pi k t + \sin 2\pi k s \sin 2\pi k t \right\} ds\, dt \qquad [104]$$

$$= \int_0^1 \mathcal{F}(s) \cos 2\pi k s\, ds \int_0^1 \mathcal{F}(t) \cos 2\pi k t\, dt$$

$$+ \int_0^1 \mathcal{F}(s) \sin 2\pi k s\, ds \int_0^1 \mathcal{F}(t) \sin 2\pi k t\, dt \qquad [105]$$

$$A_k A_{-k} = \int_{s=0}^{1} \mathcal{F}(s)\, \frac{d(\sin 2\pi k s)}{2\pi k} \int_{t=0}^{1} \mathcal{F}(t)\, \frac{d(\sin 2\pi k t)}{2\pi k}$$

$$+ \int_{s=0}^{1} \mathcal{F}(s)\, \frac{d(\cos 2\pi k s)}{2\pi k} \int_{t=0}^{1} \mathcal{F}(t)\, \frac{d(\cos 2\pi k t)}{2\pi k} \qquad [106]$$

$$= \int_{0}^{1} \frac{\sin 2\pi k s}{2\pi k} f(s)\, ds \int_{0}^{1} \frac{\sin 2\pi k t}{2\pi k} f(t)\, dt$$

$$\int_{0}^{1} \frac{\cos 2\pi k s}{2\pi k} f(s)\, ds \int_{0}^{1} \frac{\cos 2\pi k t}{2\pi k} f(t)\, dt \qquad [107]$$

$$= \int_{0}^{1}\int_{0}^{1} f(s)\, f(t)\, \frac{\cos\{2\pi k (s-t)\}}{(2\pi k)^2}\, ds\, dt \qquad [108]$$

$$\therefore\ \overline{\tilde{g}}_2(\infty) = 6\Lambda (\sin 2\beta) \int_{0}^{1}\int_{0}^{1} f(s) f(t) \sum_{k=1}^{\infty} \frac{\cos\{2\pi k (s-t)\}}{2\pi k}\, ds\, dt$$

$$+ \overline{\tilde{g}}_2(0) \qquad [109]$$

The series in eq. [109] can be put in closed form with the use of

$$\frac{1}{2} \ln \left\{ \frac{1}{2(1-\cos x)} \right\} = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n} \quad ; \quad -2\pi < x < 2\pi \qquad [110]$$

Thus:

$$\sum_{K=1}^{\infty} \frac{\cos\{2\pi K(s-t)\}}{2\pi K} = \frac{1}{4\pi} \ln\left\{\frac{1}{2\{1-\cos 2\pi(s-t)\}}\right\}; \quad |s-t| < 1 \qquad [111]$$

$$= -\frac{1}{4\pi} \ln\{2\sin^2\pi(s-t)\} \qquad [112]$$

$$= \frac{1}{2\pi} \ln\left\{\frac{1}{\sqrt{2}\,\sin\pi|s-t|}\right\} \qquad [113]$$

The limits of applicability of the expansion are adequate for the intervals of integration.

$$\therefore\ \overline{\tilde{q}}_2(\infty) = \overline{\tilde{q}}_2(0) + 6\Lambda(\sin 2\beta)\int_0^1\int_0^1 f(s)f(t)\frac{1}{2\pi}\ln\left\{\frac{1}{\sin\pi|s-t|}\right\} ds\,dt \qquad [114]$$

Here $(\ln\sqrt{2})/2$ has been eliminated because $\int_0^1\int_0^1 f(s)f(t)\,ds\,dt = 0$.

<u>Finally</u>:

$$\alpha(0) = -\overline{\tilde{q}}(0) + \delta^2 6\Lambda(\sin 2\beta)\int_0^1\int_0^1 f(s)f(t)\,\frac{\ln\{\sin\pi|s-t|\}}{2\pi}\,ds\,dt \qquad [115]$$

<u>Special Cases</u>: <u>Sinusoidal and Rectangular Ridge-Groove Geometries</u>

The foregoing formula permits ready numerical integration for arbitrary, though small, film-thickness distributions of the form:

$$\left.\begin{aligned} H = 1 + \delta f(\zeta)\ ;\ f(1+\zeta) = f(\zeta) \\ \bar{f} = f(0) = 0 \end{aligned}\right\} \qquad [116]$$

However, for two particular distributions, at least, evaluation of $\alpha(0)$ is more readily accomplished directly.

When $f(\zeta) = \sin 2\pi\zeta = \frac{1}{2i}\left\{e^{2\pi i\zeta} - e^{-2\pi i\zeta}\right\}$ [117]

the case is that treated by Wildmann. Here:

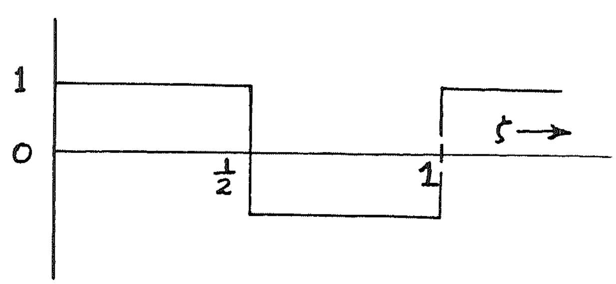
$$\frac{1}{2i}\left(e^{2\pi i\zeta} - e^{-2\pi i\zeta}\right) = \sum_{K=-\infty}^{\infty}(2\pi i K) A_K e^{2\pi i K\zeta} \qquad [118]$$

Or:

$$A_1 = A_{-1} = \frac{1}{4\pi} \quad \text{and} \quad A_K = 0 \quad \text{for } |K| > 1 .$$

$$\overline{\tilde{q}}_2(\infty) = \overline{\tilde{q}}_2(0) + \Lambda\left(\frac{3}{4\pi}\right)(\sin 2\beta) \qquad [119]$$

If $f(\zeta)$ is the repetitive jump function shown below, it is easy to obtain the desired



$$f(\zeta) = \frac{\sin(2\pi\zeta)}{\sin|2\pi\zeta|}$$

result by carrying out in eq. [107] a second integration by parts. Thus:

$$A_k A_{-k} = \int_{s=0}^{1} f(s)\, \frac{d\cos(2\pi k s)}{(2\pi k)^2} \int_{0}^{1} f(t)\, \frac{d\cos(2\pi k t)}{(2\pi k)^2}$$

$$+ \int_{s=0}^{1} f(s)\, \frac{d\sin(2\pi k s)}{(2\pi k)^2} \int_{t=0}^{1} f(t)\, d\,\frac{\sin(2\pi k t)}{(2\pi k)^2} \qquad [120]$$

Since $f(0) = f(1) = 0$,

$$A_k A_{-k} = \int_{0}^{1} \frac{\cos(2\pi k s)}{(2\pi k)^2} f'(s)\, ds \int_{0}^{1} \frac{\cos(2\pi k t)}{(2\pi k)^2} f'(t)\, dt$$

$$+ \int_{0}^{1} \frac{\sin(2\pi k s)}{(2\pi k)^2} f'(s)\, ds \int_{0}^{1} \frac{\sin(2\pi k t)}{(2\pi k)^2} f'(t)\, dt \qquad [121]$$

The derivative $f'(\zeta)$ is zero except at $\zeta = 0^+, \frac{1}{2}$ and 1^-, where it has the values $\delta(0^+)$, $-2\,\delta(\frac{1}{2})$ and $\delta(1^-)$ respectively.

$$\therefore \quad A_k A_{-k} = \frac{1}{(2\pi k)^4}\big(2 - 2\cos(\pi k)\big)\big(2 - 2\cos(\pi k)\big) \qquad [122]$$

Then: $$\overline{\tilde{q}_2}(\infty) = 6\Lambda(\sin 2\beta)\sum_{K=1}^{\infty} \frac{4(1-\cos\pi K)^2}{(2\pi K)^3} + \overline{\tilde{q}_2}(0) \quad [123]$$

$$= \frac{12\Lambda}{\pi^3}\left\{\sum_{K=1,3,5\cdot} \frac{1}{K^3}\right\}(\sin 2\beta) + \overline{\tilde{q}_2}(0) \quad [124]$$

For both of the special cases just treated, $\overline{\tilde{q}_2}(0) = 0$. In fact, this result holds for any $f(\zeta)$ satisfying the conditions in eq. [116] and anti-symmetric about ζ = 1/2. Thus (by eq. [72]):

$$\overline{\tilde{q}_2}(0) = 6\Lambda\int_0^1\int_0^\zeta\left(f^2(\zeta) - \overline{f^2}\right)d\zeta d\zeta = 6\Lambda\int_0^1(1-\zeta)\left(f^2-\overline{f^2}\right)d\zeta$$

$$= 6\Lambda\int_0^1 \tfrac{1}{2}\left(f^2-\overline{f^2}\right)d\zeta + 6\Lambda\int_0^1\left(\tfrac{1}{2}-\zeta\right)\left(f^2-\overline{f^2}\right)d\zeta \quad [125]$$

The first integral obviously vanishes. In the second integral, we note that if "f" is antisymmetric about ζ = 1/2, $f^2 - \overline{f^2}$ is symmetric and the integral vanishes. Consequently, the asserted result is proved, and, in such cases:

$$\overline{\tilde{q}(0)} = -2\Lambda\delta\int_0^1(1-\zeta)f\,d\zeta \quad [126]$$

Load Capacities With Slight Grooving:

It is now possible to write a general expression for the load-carrying capacity of any herringbone thrust plate having the film-thickness distribution given in eq. [67]. For zero flow, with $\Lambda_{eff} = 0(\delta^2)$, it is observed that eq. [31] gives:

$$\frac{d\alpha}{d\xi} = O(\delta^3) \qquad [127]$$

Hence $\alpha(\xi)$ is sensibly constant.

$$\therefore \quad \overline{\pi_1} = -2\Lambda\overline{\mathcal{F}} + 6\Lambda\overline{\mathcal{G}} + \alpha(0) \qquad [128]$$

$$= \frac{3}{\pi}\Lambda(\sin 2\beta)\,\delta^2\int_0^1\int_0^1 f(s)f(t)\ln\{\sin\pi|s-t|\}\,ds\,dt \qquad [129]$$

Very directly, as well:

$$\pi_0 = 1 + \Lambda_{eff}\,\xi = 1 + \delta^2(3\Lambda\cos\beta)\overline{f^2}\,\xi \qquad [130]$$

$$\therefore \quad W = \int_0^1(\overline{\pi}-1)\,d\xi$$

$$= \frac{3}{2}\delta^2\Lambda\cos\beta\,\overline{f^2} + \frac{3\epsilon}{\pi}\delta^2\Lambda(\sin 2\beta)\int_0^1\int_0^1 f(s)f(t)\ln\{\sin\pi|s-t|\}\,ds\,dt \qquad [131]$$

Or:

$$\frac{W}{\Lambda\delta^2} = \frac{3}{2}(\cos\beta)\overline{f^2} + \frac{3}{\pi}\epsilon(\sin 2\beta)\int_0^1\int_0^1 f(s)f(t)\ln|\sin\pi(s-t)|\,ds\,dt \qquad [132]$$

SUMMARY AND DISCUSSION OF RESULTS

The analysis required to reach useful conclusions has been fairly extensive, so it will be helpful to assemble the principal results in one place, and to discuss them thereafter.

When the film thickness function is $H(\zeta)$, an expansion in the form $\pi = \pi_0(\zeta, \xi) + \epsilon \pi_1(\zeta, \xi)$ leads to:

(a)

$$\frac{d\pi_0}{d\xi} = \Lambda_{eff} - \frac{\dot{m}_0}{2\pi_0 \overline{Q}} \qquad [133]$$

with

$$\Lambda_{eff} = \frac{\Lambda \cos\beta \left(\overline{H}\,\overline{H^{-3}} - \overline{H^{-2}}\right)}{\cos^2\beta + \sin^2\beta\, \overline{H^3}\,\overline{H^{-3}}} \qquad [134]$$

and with $\overline{Q}$, a film-thickness integral and $\dot{m}_0$, the zeroth-order mass throughput.

In addition:

(b)

$$\pi_1 = \left\{ \frac{\Lambda}{\overline{H^{-3}}} F(\zeta) + \frac{\cos\beta}{\overline{H^{-3}}} \frac{d\pi_0}{d\xi} G(\zeta) \right\} + \alpha(\xi) \qquad [135]$$

Here the film-thickness terms in curly brackets contribute the pressure "ripple", and $\alpha(\xi)$ provides a correction to the general pressure level. $\alpha(\xi)$ is found from the differential equation:

$$\frac{d}{d\xi}(\Pi_0\alpha) = \Lambda_{eff}\,\alpha - \frac{\Lambda}{\overline{H^{-3}}}\frac{d\Pi_0}{d\xi}\left[\int_0^1 \frac{QF\,d\zeta}{2\overline{Q}} + \cos^2\beta_0 \frac{\int_0^1 (H^{-3} - \overline{H^{-3}}) \int_0^{\zeta} (\overline{H^{-2}} - H^{-2})\,d\zeta\,d\zeta}{\overline{Q}}\right]$$
$$+ \frac{\Lambda\Lambda_{eff}}{\overline{H^{-3}}}\int_0^1 \frac{RF\,d\zeta}{2\overline{R}} - \frac{\dot{m}_1}{2\overline{Q}} \qquad [136]$$

The first-order correction to the mass flow, $\dot{m}_1$, is either specified, or found through satisfaction of boundary conditions for $\alpha(0)$ and $\alpha(1)$.

Reconciliation of the Π_0, Π_1 expansion with boundary conditions of constant pressure involves solution of an edge problem in the variables (η, ζ)

(c)
$$\frac{\partial}{\partial\zeta}H^3\frac{\partial\tilde{q}}{\partial\zeta} + \frac{\partial}{\partial\eta}H^3\frac{\partial\tilde{q}}{\partial\eta} + 2(\cos\beta)H^3\frac{\partial^2\tilde{q}}{\partial\zeta\partial\eta} + (\cos\beta)\frac{\partial\tilde{q}}{\partial\eta}\frac{\partial H^3}{\partial\zeta} = 0 \qquad [137]$$

with

$$\tilde{q}(\zeta,0) = \left\{\frac{\Lambda}{\overline{H^{-3}}}F(\zeta) + \frac{\cos\beta}{\overline{H^{-3}}}\frac{d\Pi_0}{d\xi}G(\zeta)\right\} \qquad [138]$$

$\tilde{q} \rightarrow$ constant (as $\eta \rightarrow \infty$) $\rightarrow \tilde{q}_\infty$.

The constant $\alpha(0)$ equals $-\tilde{q}_\infty$.

For the case of rectangular grooves and ridges (or loads), numerical results for $2(\tilde{q}_\infty - 1/2)$ have been obtained (See Figs. 4 & 5).

A small perturbation analysis in δ, when

$H = 1 + \delta f(\zeta)$, yields the following general result for the load-carrying capacity of a herringbone thrust bearing.

Thus:

(d)

$$\frac{W}{\Lambda \delta^2} = \frac{3}{2}(\cos\beta)\overline{f^2} + \epsilon \frac{3}{\pi}(\sin 2\beta)\int_0^1\int_0^1 f(s) f(t) \ln\{\sin\pi |s-t|\} \, ds\, dt \quad [139]$$

$$= \frac{3}{2}(\cos\beta) - \epsilon(0.477)(\sin 2\beta) \; ; \quad f(\zeta) = \sqrt{2} \sin 2\pi\zeta \quad [140]$$

$$= \frac{3}{2}(\cos\beta) - \epsilon(0.406)(\sin 2\beta) \; ; \quad f(\zeta) = \frac{\sin 2\pi\zeta}{|\sin 2\pi\zeta|} \quad [141]$$

It is possible to define a shifted mean pressure

(e) $$\pi^* = \overline{\pi} + \epsilon \Lambda C_1$$

which is correct through $O(\epsilon)$ and which obeys a generalized Whipple equation. Thus:

$$\frac{d\pi^*}{d\xi} = \Lambda_{eff} - \frac{\dot{m}^*}{2\pi^* \overline{Q}} \quad [142]$$

with: $$\dot{m}^* = \dot{m} - \epsilon \, 2\overline{Q}\Lambda\Lambda_{eff}\left(\frac{C_2}{H^{-3}} - C_1\right) \quad [143]$$

Typically, π^* satisfies homogeneous boundary conditions of the form:

$$\pi^* = \pi_{amb} + \epsilon \left[\text{const}\, \Lambda + \text{const} \frac{d\pi^*}{d\xi} + \text{const} \right] \quad [144]$$

Equation [133] is precisely in the form given by Whipple, but obtained in this report by a systematic expansion process that permits successive refinement of the pressure distribution. The expression for the effective lambda is noteworthy in that it is valid for arbitrary $H(\zeta)$. No generality is lost in taking $\overline{H} = 1$ (tantamount to selecting "c"). Therefore, it is seen that three parameters totally determine the major contribution to the load, π_o. These are:

$$\overline{H^{-2}},\ \overline{H^{-3}} \text{ and } \overline{H^{3}}$$

When the excursion of $H(\zeta)$ from unity is everywhere small (i.e., $|\delta f(\zeta)| << 1$, Λ_{eff} reduces to:

$$\Lambda_{eff} = 3\Lambda(\cos\beta)\,\delta^2\,\overline{f^2} \quad [145]$$

and only <u>one</u> parameter, the rms deviation $\delta\sqrt{\overline{f^2}}$, influences the major pumping action. Incidentally, this result provides a proof of the fact that the pressure-augmentation in herringbone bearings is of second order in the film-thickness deviation. Also, relative insensitivity to the character of the groove and ridge shape can be inferred.

Under conditions of no flow:

$$\Pi_0 = 1 + \Lambda_{eff}\,\xi \qquad [146]$$

and the Whipple linear pressure profile is obtained. The optimum groove angle for pressure generation is found by nothing that Λ is proportional to $\sin\beta$ and differentiating eq. [134]. The optimum angle must satisfy the following equation:

$$\tan\beta = \frac{1}{\sqrt{\overline{H^3}\,\overline{H^{-3}}}} \qquad [147]$$

and is seen to depend on a sin gle film-thickness parameter, <u>regardless</u> of film shape. The fact that β_{opt} is less than 45° for all film shapes is guaranteed by Schwarz's inequality. Thus:

$$\left[\int_0^1 H^{3/2} H^{-3/2} d\zeta\right]^2 \leq \left(\int_0^1 H^3 d\zeta\right)\left(\int_0^1 H^{-3} d\zeta\right) \qquad [148]$$

so that:

$$\overline{H^3}\,\overline{H^{-3}} \geq 1 \qquad [149]$$

and the rhs of eq. [147] is always less than, or equal to, unity.

The foregoing discussion has concerned the zeroth-order pressure, Π_0. This dominant pressure term is "smooth" in the single independent variable, ξ. Of next importance to Π_0, is the first-order pressure correction, $\Pi_1(\zeta, \xi)$. For the special case of rectangular groove-ridge geometry, both "ripple" functions $F(\zeta)$ and $G(\zeta)$ become the saw-tooth function, i.e., the transverse pressure profiles become linear, as assumed by Whipple and others. However, for other geometries, the profiles are not linear, and do not become linear as $\epsilon \to 0$.

The differential equation for the $\alpha(\xi)$-term of $\Pi_1(\zeta,\xi)$ is linear, but quite complicated. However, in the case of no flow, it simplifies as follows.

$$\Pi_0 \frac{d\alpha}{d\xi} = \Lambda \Lambda_{eff} \mathcal{K} \qquad [150]$$

where $$\mathcal{K} \equiv \frac{1}{\overline{H^{-3}}}\left[\int_0^1 \frac{RF\,d\zeta}{2\overline{R}} - \int_0^1 \frac{QF\,d\zeta}{2\overline{Q}} - \frac{\cos^2\beta}{\overline{Q}}\int_0^1 (H^{-3}-\overline{H^{-3}})\int_0^{\zeta}(H^{-2}-\overline{H^{-2}})\,d\zeta\,d\zeta\right] \qquad [151]$$

$$\therefore \alpha = \Lambda \mathcal{K} \ln(1+\Lambda_{eff}\xi) + \alpha(0) \qquad [152]$$

Here the general case shows a logarithmic behavior in Λ . However, the important case of rectangular geometry again is exceptional, since, as shown in Appendix C, $\mathcal{K} = 0$. Whereupon $\alpha(\xi) = \alpha(0) =$ constant.

Equations [137] and [138] set the problem for a residual function, $\tilde{g}$, permitting asymptotic matching between the interior first-order ripple function, $\Pi_1(\zeta,\xi)$, and boundary condition of ambient pressure, which requires all ripples to be absent. This problem is to be regarded only as typical of those that can arise at bearing edges. It is, however, a generalization of Muijdermann's edge-effect treatment, extended by the present analysis to bearings with compressible fluid and variable groove geometry. Terminal values of $\alpha(\xi)$ are provided by residual $\tilde{g}$-functions. These values are joined by solutions of the α-differential equation --- non-constant solutions except in special circumstances.

A important exception is that of <u>rectangular grooving</u> at <u>no flow</u>.

The load-carrying capacity of herringbone thrust plates with shallow modulations is given by eq. [139]. This general result can be used to explore the effects of variations of the type of grooving, although if the two particular cases cited are indicative, the edge effects for a given rms film thickness excursion will be insensitive to shape. Note that the edge effect with sinusoidal grooving is just 17.5% greater than for rectangular.

Finally, the "complete" differential equation given by eq. [142] is to be noted. The fact that, for $H = H(\zeta) = H(1+\zeta)$, zero and first-order effects can be incorporated in a single ordinary differential equation gives one reason to hope that a generalization to near-periodic film thicknesses and curved grooving may perhaps be possible by means of a single partial differential equation of the type obtained by Vohr and Pan,

APPENDIX A

FILM-THICKNESS INTEGRALS

Define: $$F(\zeta) \equiv 2\int_0^{\zeta}\left(H^{-3}\,\overline{H^{-2}} - \overline{H^{-3}}\,H^{-2}\right)d\zeta \qquad [153]$$

$$G(\zeta) \equiv \int_0^{\zeta}\left(H^{-3} - \overline{H^{-3}}\right)d\zeta \qquad [154]$$

$$Q(\zeta) \equiv \overline{H^{-3}}\,H^{3} + \cos^2\beta\left(1 - \overline{H^{-3}}\,H^{3}\right) \qquad [155]$$

$$R(\zeta) \equiv H^{-2} - H\,\overline{H^{-3}} \qquad [156]$$

Then:

$$\overline{H^{-3}}\,\psi_1 = \Lambda\sqrt{\psi_0}\,F(\zeta) + (\cos\beta)\frac{d\psi_0}{d\xi}\,G(\zeta) + 2\,\overline{H^{-3}}\,\alpha(\xi)\sqrt{\psi_0} \qquad [157]$$

$$K_1 = \int_0^1 H^3 F\,d\zeta \qquad K_2 = \int_0^1 H^3 G\,d\zeta$$

$$K_3 = \int_0^1 R F\,d\zeta \qquad K_4 = \int_0^1 R G\,d\zeta$$

$$K_5 = \int_0^1 G'(\zeta)\int_0^{\zeta} H^3\,\overline{H^{-3}}\,d\zeta\,d\zeta \qquad K_6 = \int_0^1 \left(1 - H^3\,\overline{H^{-3}}\right)F\,d\zeta$$

$$K_7 = \int_0^1 \left(1 - H^3\,\overline{H^{-3}}\right)G\,d\zeta \qquad K_8 = \int_0^1 \left(1 - H^3\,\overline{H^{-3}}\right)d\zeta$$

$$K_9 = \int_0^1 G'(\zeta)\int_0^{\zeta} H^3 F'(\zeta)\,d\zeta\,d\zeta \qquad K_{10} = \int_0^1 G'(\zeta)\int_0^{\zeta} H^3 G'(\zeta)\,d\zeta\,d\zeta$$

Consider the grouping $K_5 + \overline{H^{-3}}K_2$ which appears in the coefficient of $\frac{d^2\psi_0}{d\xi^2}$ in eq. [30]. Thus:

With

$$\mathcal{F} = \int_0^\zeta H^3 \overline{H^{-3}}\, d\zeta \qquad [158]$$

this equation becomes:

$$K_5 + \overline{H^{-3}} K_2 = \int_0^1 (G'\mathcal{F} + G\mathcal{F}')\, d\zeta = G(1)\mathcal{F}(1) - G(0)\mathcal{F}(0) = 0 \qquad [159]$$

Now consider the grouping:

$$K_7 + K_{10} = \int_0^1 \left(\{1 - H^3\overline{H^{-3}}\} G + G' \int_0^\zeta \{1 - H^3 \overline{H^{-3}}\}\, d\zeta \right) d\zeta \qquad [160]$$

= 0 by similar arguments.

Therefore, the coefficient of $\frac{d^2\psi_0}{d\xi^2}$ vanishes identically.

The coefficient of $\Lambda \frac{d\pi_0}{d\xi}$ is:

$$\frac{1}{\overline{H^{-3}}} \left[\overline{H^{-3}} K_1 + \cos^2\beta \{K_6 + 2K_4 + K_9\} \right] =$$

$$\frac{1}{\overline{H^{-3}}} \left[\int_0^1 H^3 \overline{H^{-3}} F\, d\zeta + \cos^2\beta \int_0^1 (1 - H^3\overline{H^{-3}}) F\, d\zeta + \cos^2\beta \left\{ 2\int_0^1 RG\, d\zeta + 2\int_0^1 G'(\zeta) \int_0^\zeta (\overline{H^{-2}} - H\overline{H^{-3}})\, d\zeta\, d\zeta \right. \right]$$

$$= \frac{1}{\overline{H^{-3}}} \left[\int_0^1 QF\, d\zeta + 2\cos^2\beta \int_0^1 (H^{-3} - \overline{H^{-3}}) \int_0^\zeta (\overline{H^{-2}} - H^{-2})\, d\zeta\, d\zeta \right] \qquad [161]$$

$$\Lambda_{eff} = -\frac{\Lambda(\cos\beta)\overline{R}}{\overline{Q}} \qquad [162]$$

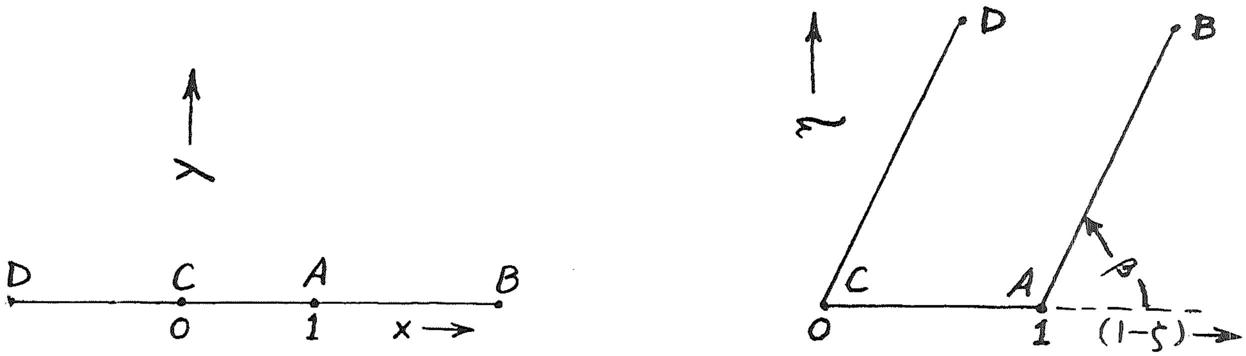
APPENDIX B

SINGLE-GROOVE EDGE CORRECTION

The mapping function

$$w = \frac{\sin\beta}{\pi}\int_0^z z^{\frac{\beta}{\pi}-1}(z-1)^{-\frac{\beta}{\pi}}e^{i\beta}\,dz \equiv 1-\zeta+i\eta \qquad [163]$$

converts the real axis of the z-plane into the form shown below in the w-plane



z-plane $z = x + iy$

w-plane $w = (1-\zeta) + i\eta$

When $\beta = \frac{\pi}{2}$ we take $1-\zeta \equiv u, \ \eta \equiv v$

Along $y = o$

$$1-\zeta = \frac{\sin\beta}{\pi}\int_0^x x^{\frac{\beta}{\pi}-1}(1-x)^{-\frac{\beta}{\pi}}\,dx\,; \qquad u = \frac{2}{\pi}\tan^{-1}\sqrt{\frac{x}{1-x}} \qquad [164]$$

Now in the (u, v) plane

$$\tilde{q}_\infty = \int_0^1 \hat{q}(u)\,du = \tilde{q}(1) - \int_0^1 u \frac{d\tilde{q}}{du}\,du \qquad [165]$$

$$= 1 - \int_0^1 u \frac{d\tilde{q}}{du}\,du \qquad [166]$$

$$= 1 - \frac{2}{\pi}\int_0^1 \tan^{-1}\sqrt{\frac{x}{1-x}}\ \frac{\sin\beta}{\pi}\, x^{\frac{\beta}{\pi}-1}(1-x)^{-\frac{\beta}{\pi}}\,dx \qquad [167]$$

[Note: $\hat{q} = 1-\zeta$].

Next let $x = \dfrac{t^2}{1+t^2}$; $t = \sqrt{\dfrac{x}{1-x}}$ [168]

then:

$$\tilde{q}_\infty = 1 - \frac{2(\sin\beta)}{\pi^2}\int_0^\infty \tan^{-1}(t)\ t^{\frac{2\beta}{\pi}-1}\,\frac{2\,dt}{1+t^2} \qquad [169]$$

Write $\tan^{-1}(t)$ in the following integral form. Thus:

$$\tan^{-1}(t) = t\int_0^1 \frac{ds}{1+t^2s^2} \qquad [170]$$

$$\therefore\ \tilde{q}_\infty = 1 - \frac{2(\sin\beta)}{\pi^2}\int_{s=0}^{1} 2\int_{t=0}^{\infty} \frac{t^{\frac{2\beta}{\pi}}\,dt}{(1+t^2s^2)(1+t^2)}\,ds \qquad [171]$$

Resolution into partial fractions gives:

$$\tilde{q}_\infty = 1 - \frac{\sin\beta}{\pi^2}\int_{s=0}^{1} 2\int_0^\infty \left\{ \frac{1}{(1-s^2)}\frac{1}{(1+t^2)} - \frac{s^2}{1-s^2}\frac{1}{1+t^2s^2} \right\} t^{\frac{2\beta}{\pi}}\, dt\, ds \qquad [172]$$

Use is made of the infinite integral

$$\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad [173]$$

and the reflection formula for the Gamma function:

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin(n\pi)} \qquad [174]$$

Then:

$$\tilde{q}_\infty = 1 - 2\frac{\tan\beta}{\pi}\int_0^1 \frac{(1-s^m)}{(1-s^2)}\, ds \; ; \quad m = 1 - \frac{2\beta}{\pi} \qquad [175]$$

The above definite integral converges, but not with the terms 1, s^m treated separately. Accordingly, the behavior of an integral with the denominator "shaded" to $(1-s^2)^{\frac{1-n}{2}}$ will first be examined.

$$I \equiv \int_0^1 \frac{(1-s^m)}{(1-s^2)^{\frac{1-n}{2}}}\, ds = \frac{1}{2}\left[B\left(\tfrac{1}{2}, \tfrac{n+1}{2}\right) - B\left(\tfrac{m+1}{2}, \tfrac{n+1}{2}\right)\right] \qquad [176]$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{2}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}+\frac{m+1}{2}\right)}\right] \qquad [177]$$

Let $\theta = \frac{n+1}{2}$, and examine "I" as $\theta \to 0$.

$$I = \frac{1}{2} \frac{\theta\,\Gamma(\theta)}{\Gamma(\theta+\frac{1}{2})\Gamma(\theta+\frac{m+1}{2})}\left[\frac{\Gamma(\frac{1}{2})\Gamma(\theta+\frac{m+1}{2}) - \Gamma(\frac{m+1}{2})\Gamma(\theta+\frac{1}{2})}{\theta}\right] \quad [178]$$

$$\longrightarrow \frac{1}{2} \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{m+1}{2})} \left[\Gamma(\tfrac{1}{2})\Gamma'(\tfrac{m+1}{2}) - \Gamma(\tfrac{m+1}{2})\Gamma'(\tfrac{1}{2})\right] \quad [179]$$

Hence:

$$\tilde{q}_\infty = 1 - \frac{\tan\beta}{\pi}\left[\frac{\Gamma'(1-\frac{\beta}{\pi})}{\Gamma(1-\frac{\beta}{\pi})} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})}\right] \quad [180]$$

$$1 - \frac{\tan\beta}{\pi}\left\{\psi(1-\tfrac{\beta}{\pi}) - \psi(\tfrac{1}{2})\right\} \quad [181]$$

More conveniently for computation:

$$\tilde{q}_\infty = 1 - \frac{\tan\beta}{\pi}\left\{\psi(2-\tfrac{\beta}{\pi}) - \psi(\tfrac{1}{2}) - \frac{1}{(1-\frac{\beta}{\pi})}\right\} \quad [182]$$

APPENDIX C

SPECIAL RELATIONS FOR RECTANGULAR GROOVING

Important simplifications occur in the film-thickness integrals for rectangular grooving; i. e., for grooving such that:

$$\left.\begin{aligned} H(\zeta) &= H_1 \; ; \; 0 \leq \zeta < r \\ H(\zeta) &= H_2 \; ; \; r < \zeta \leq 1 \end{aligned}\right\} \qquad [183]$$

Consider the integral:

$$F(\zeta) = 2\int_0^1 \left(H^{-3}\overline{H^{-2}} - \overline{H^{-3}}H^{-2}\right) d\zeta \qquad [184]$$

For $0 \leq \zeta \leq r$:

$$F(\zeta) = 2\left(H_1^{-3}\overline{H^{-2}} - \overline{H^{-3}}H_1^{-2}\right)\zeta \qquad [185]$$

But also, beyond "r", $F(\zeta)$ is linear in "ζ", decreasing to zero at $\zeta = 1$. Therefore, for $r \leq \zeta \leq 1$.

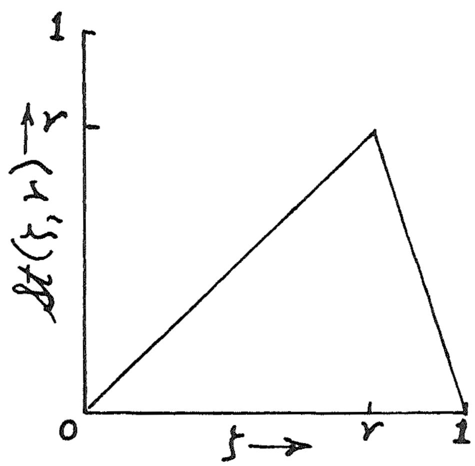
$$F(\zeta) = 2\left(H_1^{-3}\overline{H^{-2}} - \overline{H^{-3}}H_1^{-2}\right) r - 2\left(H_1^{-3}\overline{H^{-2}} - \overline{H^{-3}}H_1^{-2}\right) r \frac{(\zeta - r)}{(1-\zeta)} \qquad [186]$$

$$= 2\left(H_1^{-3}\overline{H^{-2}} - \overline{H^{-3}}H_1^{-2}\right) \frac{r}{1-r} (1-\zeta) \qquad [187]$$

For all "ζ", then:

$$F(\zeta) = 2\left(H_1^{-3}\overline{H^{-2}} - \overline{H^{-3}}H_1^{-2}\right) St(\zeta, r) \qquad [188]$$

where $St(\zeta, r)$ is the saw-tooth function depicted below.



Similarly,

$$G(\zeta) = (H_1^{-3} - \overline{H^{-3}})\,\delta t(\zeta, r) \qquad [189]$$

$$\int_0^{\zeta} (\overline{H^{-2}} - H^{-2})\,d\zeta = (\overline{H^{-2}} - H_1^{-2})\,\delta t(\zeta, r) \qquad [190]$$

Now it is important to observe that if $f(\zeta)$ is any function with constant (though possibly different) values in the two intervals $0 \leq \zeta < r$, $r < \zeta \leq 1$ integration against $\delta t(\zeta, r)$ yields $r\overline{f}/2$.

$$\therefore \int_0^1 Q F\,d\zeta = (r\overline{Q})\,2(H_1^{-3}\overline{H^{-2}} - \overline{H^{-3}}H_1^{-2})/2\text{, etc.} \qquad [191]$$

In particular:

$$\frac{\int_0^1 RF\,d\zeta}{2\overline{R}} = \frac{\int_0^1 QF\,d\zeta}{2\overline{Q}} \qquad [192]$$

and:

$$\int_0^1 (H^{-3} - \overline{H^{-3}}) \int_0^{\zeta} (\overline{H^{-2}} - H^{-2})\,d\zeta\,d\zeta = 0 \qquad [193]$$

Hence "$\mathcal{K}$" defined in eq. 151 is zero.

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A narrow-groove theory for gas on liquid-lubricated herringbone thrust bearings is developed by means of two matched asymptotic expansions. The first expansion, for the film interior,yields a generalized Whipple equation for the average pressure level. The second expansion, for the film edges, yields a generalized Muijdermann-Body pressure correction. Arbitrary transverse groove shape is accommodated by the analysis.

The prognosis for development along present lines of a single partial differential equation to include first-order groove-width effects, both in the film interior and at its edges, is very good.

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